Symmetries and particles

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Relativistic quantum theory (Berestetskij *et al.*, 1991) ...: We describe a pointwise, structureless, elementary, free particle by a finite dimensional irreducible unitary representation of its group symmetries (the Galileo group in the non-relativistic case and the Poincaré group in the relativistic case, extended to the parity transformation). The invariants of the group are the mass and the spin. The wave functions of the particles are in bijective correspondence with the vectors of such representations, and the scalar product for such vectors is expressible in terms of wave functions. We determine the wave equation satisfied by the particles. In the relativistic case, the locality requirement, forces the introduction of "negative energy" solutions. It is an experimental fact that the number of particles may change in physical processes. Then, there exist transitions between states with different number of particles. We will present a formalism that allows to describe systems of many free particles, used in any many-body theory, relativistic or not, and known as Fock method. It allows to describe many particles states with the correct statistics and to introduce operators that change the number of particles (creation and annihilation operators). We will introduce the free field operators, and we will interpret in terms of field operators the negative energy solutions of the equations of free motion. We will denote as "antiparticles" the negative energy particles with a non-hermitian field operator. We construct the representation of the group on the many free particles states. And we prove the spin-statistics theorem which states that, as a consequence of Lorentz invariance and of locality, half integer spin particles must obey to Fermi statistics and integer spin particles must obey to Bose statistics.

CONTENTS

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I. DEFINITION OF INVARIANCE

A reference frame is defined by a set of operative rules to measure physical quantities.

The same physical phenomenon can be observed from two different reference frames. In order for the two reference frames to be defined, the transformation between the quantities measured in the two frames must be known.

In a given reference frame a phenomenon obeys certain physical laws. A physical law is a relationship which poses conditions on the quantities measured at a given instant.

The frames are said to be equivalent respect to a class of phenomena if:

a) Any physical situation realizable in one can also be realizable in the other.

b) The time evolution laws are the same in the two frames.

The equivalence between frames produced by the invariance is an equivalence relationship in the mathematical sense: Given R, R', R'' three frames; R is equivalent to R, if R is equivalent to R' then R' is equivalent to R; if R is equivalent to R' and R' is equivalent to R'' then R is equivalent to R''.

The transformation laws between quantities in equivalent frames form a group:

a) The identity transformation exists: The one between any frame and itself.

b) Given any transformation, an inverse transformation exists which is itself an equivalence relationship respect to the class of phenomena in exam.

c) The product of two equivalence relationships, defined as the application in succession and ordered of two transformations, is still an equivalence relationship.

The equivalence of a class of frames relative to a set of phenomena is called *invariance* of such phenomena relative to the group of transformations between the frames.

A. Conventions

Through the note we will conform to the following conventions:

1. Units

We will always use relativistic units with $\hbar = 1, c = 1$. In these units, we have for the elementary charge $e^2/4\pi = 1/137$.

2. Fourier transform

The tridimensional Fourier transform is

$$f(\mathbf{p}) = \int f(\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{p}}\, d\mathbf{q}, \tag{1.1}$$

$$f(\mathbf{q}) = \int f(\mathbf{p}) e^{i\mathbf{q}\cdot\mathbf{p}}\, \frac{d\mathbf{p}}{(2\pi)^3}, \tag{1.2}$$

and analogously for the four-dimensional case.

3. Operators

We will not introduce a different symbol for the operators on the Hilbert space and their eigenvalues. The reader should understand the difference from the context of the various equations introduced.

II. INVARIANCE IN QUANTUM MECHANICS

In quantum mechanics the invariance respect to a change of reference frame is defined as follows:

a) The possible states in the two frames are the vectors of a same Hilbert space. The observables are the same. The transformation law is a mapping of the Hilbert space onto itself.

b) Starting from the same initial state the time evolution is the same in the two frames.

The invariance transformations are a group. So an invariance transformation is a *realization* of the group on an Hilbert space.

Let $|a\rangle$ be a state, in a certain frame, defined by the simultaneous measure of a complete set of commuting observables. Any vector of the form $x_a|a\rangle$ where x_a is an arbitrary phase factor, is an eigenstate of the same observables with the same eigenvalues. So it represents the same physical state. The phase is not observable. A measurement on $|a\rangle$ means to observe the probability that $|a\rangle$ contains a state $|b\rangle$ defined by the measure instruments. What one measures is

$$P_{ab} = |\langle b|a\rangle|^2, \tag{2.1}$$

where the phases x_a and x_b cancel. A vector of the Hilbert space modulo a phase is called a "ray" of the Hilbert space and will be denoted $|\{a\}\rangle$.

Wigner theorem: Given a bijective transformation between rays in a Hilbert space $|\{s\}\rangle \to |\{s'\}\rangle$ such that

$$|\langle\{s_2'\}|\{s_1'\}\rangle|^2 = |\langle\{s_2\}|\{s_1\}\rangle|^2 \quad \forall|\{s_1\}\rangle, |\{s_2\}\rangle \tag{2.2}$$

it is always possible to choose the phases in such a way that the transformation is realized on the Hilbert space vectors as a unitary or antiunitary transformation.

Proof:

1. Let $|e_n\rangle$ be an orthonormal complete base of the Hilbert space and let $|\{e_n\}\rangle$ be the correspondent rays. The transformed rays are orthonormal

$$\langle e_i|e_j\rangle = \delta_{ij} \implies |\langle\{e_i'\}|\{e_j'\}\rangle|^2 = \delta_{ij} \tag{2.3}$$

Let us choose in an arbitrary way a set of phases on the rays $|\{e_i'\}\rangle$, i.e. a set of vectors $|e_i'\rangle$ that represent the states. Then

$$\langle e_i'|e_j'\rangle = \delta_{ij}, \tag{2.4}$$

The set of vectors so obtained is also a complete base of the Hilbert space. In fact, if there exists a vector $|v'\rangle$ such that $\langle v'|v'\rangle \neq 0$ and $\langle v'|e'_n\rangle = 0 \quad \forall n$, then, by hypothesis, there would exist a vector $|v\rangle$ such that $\langle v|v\rangle \neq 0$ and $\langle v|e_n\rangle = 0 \quad \forall n$, against the hypothesis of completeness of the base $|e_n\rangle$.

2. Let $|F_k\rangle = |e_1\rangle + |e_k\rangle$. The generic representative of the transformed ray $|\{F'_k\}\rangle$ will be

$$|F'_k\rangle = x_k(|e'_1\rangle + y_k|e'_k\rangle), \tag{2.5}$$

with x_k and y_k phases factors. In fact

$$|\langle F_k|e_n\rangle| = \delta_{n1} + \delta_{nk} \implies |\langle F'_k|e'_n\rangle| = \delta_{n1} + \delta_{nk}. \tag{2.6}$$

Next I can define the following S transformation

$$|Se_1\rangle \;=\; |e'_1\rangle \quad |Se_k\rangle = y_k|e'_k\rangle \tag{2.7}$$

$$|SF_k\rangle \;=\; \frac{1}{x_k}|F'_k\rangle = |e'_1\rangle + y_k|e'_k\rangle. \tag{2.8}$$

With this choice

$$|SF_k\rangle = |Se_1\rangle + |Se_k\rangle. \tag{2.9}$$

In other words we realized the transformation S as a linear transformation on vectors of kind $|F_k\rangle$. Let us next extend this construction to all vectors of the Hilbert space.

3. Consider a generic vector

$$|v\rangle = \sum_n a_n|e_n\rangle. \tag{2.10}$$

Let us assume, without loss of generality, a_1 real. The correspondent ray $|\{v\}\rangle$ will be transformed into a ray $|\{v'\}\rangle$ with the following generic representative

$$|v'\rangle = \sum_n a'_n|e'_n\rangle, \tag{2.11}$$

and since by hypothesis

$$|\langle v|e_n\rangle|^2 = |\langle v'|e'_n\rangle|^2, \tag{2.12}$$

we have

$$|a'_n| = |a_n|. \tag{2.13}$$

We define

$$|Se_1\rangle \;=\; |e'_1\rangle, \tag{2.14}$$

$$|Se_n\rangle \;=\; y_n|e'_n\rangle \quad \forall n \neq 1, \tag{2.15}$$

with y_n some phase factors, so that for any vector belonging to the transformed ray $|\{v'\}\rangle$

$$|v'\rangle = x\left\{a_1|Se_1\rangle + \sum_{n=2}^{\infty}\frac{a'_n}{y_n}|Se_n\rangle\right\}, \tag{2.16}$$

with x a phase factor. We then define

$$|Sv\rangle = \frac{1}{x}|v'\rangle. \tag{2.17}$$

By hypothesis it must be

$$|\langle F_k|v\rangle|^2 = |a_1 + a_k|^2 = |\langle SF_k|Sv\rangle|^2 = \left|a_1 + \frac{a'_k}{y_k}\right|^2. \tag{2.18}$$

Since we also have $|a_k| = |a'_k|$ we require

$$\mathbf{Re}(a_1 a_k) = \mathbf{Re}\left(a_1 \frac{a'_k}{y_k}\right). \tag{2.19}$$

Then there are only two possibilities:

i. $a_k = a'_k/y_k$
ii. $a_k = (a'_k/y_k)^*$

or

i. $|Sv\rangle = S(\sum_n a_n|e_n\rangle) = \sum_n a_n|Se_n\rangle$
ii. $|Sv\rangle = S(\sum_n a_n|e_n\rangle) = \sum_n a_n^*|Se_n\rangle$

In the first case the operator S is linear, in the second is antilinear. We also have

i. $\langle Sv_1|Sv_2\rangle = \langle v_1|v_2\rangle \quad \forall|v_1\rangle, |v_2\rangle$
ii. $\langle Sv_1|Sv_2\rangle = \langle v_2|v_1\rangle \quad \forall|v_1\rangle, |v_2\rangle$

In the first case S is unitary, in the second it is antiunitary.

III. INVARIANCE AND TIME EVOLUTION

The requirement b) for invariance tells us that the evolution of the transformed must coincide with the transformation of the evolved

$$U(t,t')S(t')|\psi\rangle = S(t)U(t,t')|\psi\rangle, \tag{3.1}$$

where $U(t,t')$ is the time evolution operator. Since $|\psi\rangle$ is arbitrary we must have

$$S^\dagger(t)U(t,t')S(t') = U(t,t'). \tag{3.2}$$

If the Hamiltonian H is independent of time

$$U(t,t') = e^{-iH(t-t')}, \tag{3.3}$$

and we require

$$S(t) = e^{-iH(t-t')}S(t')e^{iH(t-t')}. \tag{3.4}$$

IV. GALILEAN RELATIVITY

We require invariance under translations, rotations, and velocity transformations for pointwise non relativistic particles.

A. Spatial translations

Let us consider a reference frame R' translated by $\mathbf{a}$ relative to the frame R. If the spatial translations are a symmetry of the system it must exist a unitary transformation $U(\mathbf{a})$ which relates the dynamical variables $\mathbf{q}'$ and $\mathbf{p}'$ in R' to the variables $\mathbf{q}$ and $\mathbf{p}$ in R. The transformation law must be

$$\mathbf{q}' = \mathbf{q} - \mathbf{a}, \tag{4.1}$$
$$\mathbf{p}' = \mathbf{p}. \tag{4.2}$$

It is easy to see that the unitary operator exists and is

$$U(\mathbf{a}) = e^{i\mathbf{a}\cdot\mathbf{p}}. \tag{4.3}$$

Since the transformation is unitary the commutation relations do not change

$$[q'_i, p'_j] = [q_i, p_j] = i\delta_{ij}, \quad (4.4)$$
$$[q'_i, q'_j] = [q_i, q_j] = 0, \quad (4.5)$$
$$[p'_i, p'_j] = [p_i, p_j] = 0, \quad (4.6)$$

where $\mathbf{q} = U(\mathbf{a})^\dagger \mathbf{q} U(\mathbf{a})$ and $\mathbf{p} = U(\mathbf{a})^\dagger \mathbf{p} U(\mathbf{a})$. Moreover from Hadamard lemma (A11) follows immediately that Eqs. (4.1)-(4.2) are satisfied.

The invariance of the time evolution between two frames R and R' imposes

$$U^\dagger(\mathbf{a}, t) e^{-iH(t-t')} U(\mathbf{a}, t') = e^{-iH(t-t')}, \quad (4.7)$$

which means

$$[\mathbf{p}, H] = 0. \quad (4.8)$$

In other words, the momentum is a constant of motion. We can also write

$$\frac{\partial H}{\partial \mathbf{q}} = 0. \quad (4.9)$$

B. Rotations

A rotation is defined by a versor $\hat{\mathbf{n}}$ which indicates the axis of rotation and an angle θ. We define $\boldsymbol{\theta} = \theta \hat{\mathbf{n}}$. The angles are taken as positive for anti-clockwise rotations. Let us consider a frame R' rotated by $\boldsymbol{\theta}$ relative to frame R. The component of a vector $\mathbf{v}$ will change according to

$$v'_i = R(\boldsymbol{\theta})_{ij} v_j, \quad (4.10)$$

where $R(\boldsymbol{\theta})$ is the rotation matrix. For infinitesimal transformations

$$\delta \mathbf{v} = \mathbf{v}' - \mathbf{v} \approx -\boldsymbol{\theta} \wedge \mathbf{v}. \quad (4.11)$$

If the quantum system is invariant under rotations it must be possible to construct a unitary transformation on the Hilbert space which realizes the transformation and commutes with the time evolution. Let us then consider the angular momentum

$$\mathbf{J} = \mathbf{q} \wedge \mathbf{p}. \quad (4.12)$$

It is easy to verify that for $\mathbf{v} = \mathbf{q}$ or $\mathbf{v} = \mathbf{p}$ we have

$$[\boldsymbol{\theta} \cdot \mathbf{J}, \mathbf{v}] = -i\boldsymbol{\theta} \wedge \mathbf{v}. \quad (4.13)$$

Then the transformation we are looking for is

$$U(\boldsymbol{\theta}) = e^{i\boldsymbol{\theta} \cdot \mathbf{J}}, \quad (4.14)$$

as can be readily verified for infinitesimal transformations

$$\mathbf{v}' = U^\dagger(\boldsymbol{\theta}) \mathbf{v} U(\boldsymbol{\theta}) \approx \mathbf{v} - i[\boldsymbol{\theta} \cdot \mathbf{J}, \mathbf{v}] = \mathbf{v} - \boldsymbol{\theta} \wedge \mathbf{v}. \quad (4.15)$$

The transformation commutes with the time evolution if

$$[\mathbf{J}, H] = 0 \quad (4.16)$$

which means that H must be a scalar and the angular momentum a constant of motion. Since the transformation is unitary it preserves the commutation relations.

If the particle has a spin the generator of the rotations is the total angular momentum

$$\mathbf{J} = \mathbf{q} \wedge \mathbf{p} + \mathbf{s}. \quad (4.17)$$

C. Galilean transformations

If we go from a frame R to a frame R' moving relative to R with a constant speed $\mathbf{v}$ we must have

$$\mathbf{q}' = \mathbf{q} - t\mathbf{v}, \tag{4.18}$$
$$\mathbf{p}' = \mathbf{p} - m\mathbf{v}. \tag{4.19}$$

It is easy to verify that these laws of transformation are induced by the unitary operator

$$U(t, \mathbf{v}) = e^{i(\mathbf{p}t - \mathbf{q}m)\cdot\mathbf{v}}, \tag{4.20}$$

so that

$$U^\dagger(t, \mathbf{v})\mathbf{q}U(t, \mathbf{v}) = \mathbf{q} - t\mathbf{v}, \tag{4.21}$$
$$U^\dagger(t, \mathbf{v})\mathbf{p}U(t, \mathbf{v}) = \mathbf{p} - m\mathbf{v}. \tag{4.22}$$

If the Galilean transformation has to be an invariance we must also require

$$U(t, \mathbf{v}) = e^{-iH(t-t')}U(t', \mathbf{v})e^{iH(t-t')}, \tag{4.23}$$

or

$$t\mathbf{p} - m\mathbf{q} = e^{-iH(t-t')}(t'\mathbf{p} - m\mathbf{q})e^{iH(t-t')}. \tag{4.24}$$

If the system is invariant under translations $[\mathbf{p}, H] = 0$, so

$$(t - t')\mathbf{p} = m\mathbf{q} - me^{-iH(t-t')}\mathbf{q}e^{iH(t-t')}. \tag{4.25}$$

For infinitesimal time differences we get

$$\frac{\mathbf{p}}{m} = i[H, \mathbf{q}] = \frac{\partial H}{\partial \mathbf{p}}. \tag{4.26}$$

So

$$H = \frac{\mathbf{p}^2}{2m}. \tag{4.27}$$

D. Galileo group

We analyzed the symmetries under translations, rotations, and Galileo transformations for a non relativistic system. The corresponding unitary transformations are

$$U(\mathbf{a}) = e^{i\mathbf{a}\cdot\mathbf{p}}, \tag{4.28}$$
$$U(\boldsymbol{\theta}) = e^{i\boldsymbol{\theta}\cdot\mathbf{J}}, \tag{4.29}$$
$$U(\mathbf{v}) = e^{-i\mathbf{v}\cdot\mathbf{K}} \quad \mathbf{K} = m\mathbf{q} - t\mathbf{p} \tag{4.30}$$

The group corresponding to the set of these transformations is called "Galileo group" and the corresponding invariance "galilean invariance".

From the canonical commutation relationships, the following algebra for the group generators, follows

$$[p_\mu, p_\nu] = 0 \quad P_0 = H \tag{4.31}$$
$$[\mathbf{J}, H] = 0 \quad [J_i, p_j] = i\epsilon_{ijk}p_k \tag{4.32}$$
$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad [J_i, K_j] = i\epsilon_{ijk}K_k \tag{4.33}$$
$$[K_i, K_j] = 0 \quad [K_i, p_j] = im\delta_{ij} \quad [K_i, H] = ip_i \tag{4.34}$$

In the Hilbert space of the physical system is then defined an unitary representation of the group that transforms the spec into itself.

If this representation is reducible it is possible to write the Hilbert space as a direct sum of one or more orthogonal Hilbert spaces each one transforming in itself. The generators are written as sum of the generators acting in each subspace and generators acting on different irreducible subspaces commute. The states in each subspace evolve with their Hamiltonian each in states belonging to the same subspace.

A physical system can then be written as a sum of irreducible representations of the Galileo group.

The simplest case is a particle without internal structure. In this case the only internal variable is the spin which commutes with the orbital variables. A complete set of state is

$$|\mathbf{p}\rangle|s, s_z\rangle. \tag{4.35}$$

Assuming the usual metric

$$\begin{aligned}\langle \mathbf{p}'|\mathbf{p}\rangle &= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}'), &(4.36)\\ \langle s_z'|s_z\rangle &= \delta_{s_z' s_z} &(4.37)\end{aligned}$$

these states constitute an irreducible representation of the Galileo group if the states $|s_z\rangle$ are an irreducible representation of internal rotations. Let us show this explicitly:

$$\begin{aligned}\mathbf{p}|\mathbf{p}\rangle &= \mathbf{p}|\mathbf{p}\rangle, &(4.38)\\ \mathbf{p}e^{i\boldsymbol{\theta}\cdot\mathbf{J}}|\mathbf{p}\rangle &= e^{i\boldsymbol{\theta}\cdot\mathbf{J}}e^{-i\boldsymbol{\theta}\cdot\mathbf{J}}\mathbf{p}e^{i\boldsymbol{\theta}\cdot\mathbf{J}}|\mathbf{p}\rangle\\ &= R(\boldsymbol{\theta})\mathbf{p}e^{i\boldsymbol{\theta}\cdot\mathbf{J}}|\mathbf{p}\rangle, &(4.39)\end{aligned}$$

where $\mathbf{p}$ on the right hand side denotes the momentum operator acting on the eigenstate $|\mathbf{p}\rangle$ and on the left denotes the eigenvalue. The eigenvalues of the rotated state is the rotated momentum. In the same way:

$$\begin{aligned}\mathbf{p}e^{-i\mathbf{v}\cdot\mathbf{K}}|\mathbf{p}\rangle &= e^{-i\mathbf{v}\cdot\mathbf{K}}e^{i\mathbf{v}\cdot\mathbf{K}}\mathbf{p}e^{-i\mathbf{v}\cdot\mathbf{K}}|\mathbf{p}\rangle\\ &= (\mathbf{p}-m\mathbf{v})e^{-i\mathbf{v}\cdot\mathbf{K}}|\mathbf{p}\rangle, &(4.40)\end{aligned}$$

so

$$\begin{aligned}e^{i\boldsymbol{\theta}\cdot\mathbf{J}}|\mathbf{p}\rangle &= |R(\boldsymbol{\theta})\mathbf{p}\rangle, &(4.41)\\ e^{-i\mathbf{v}\cdot\mathbf{K}}|\mathbf{p}\rangle &= |\mathbf{p}-m\mathbf{v}\rangle, &(4.42)\end{aligned}$$

and we see that starting from any vector $|\mathbf{p}\rangle$ it is possible to reach any other vector $|\mathbf{p}'\rangle$ through successive applications of rotations or of Galileo transformations. The internal degrees of freedom only transform by rotations independently.

So a pointwise free particle is described by an irreducible unitary representation of the Galileo group.

E. Parity invariance

The parity transformation is defined by

$$\mathbf{p}\to-\mathbf{p} \quad \mathbf{q}\to-\mathbf{q} \quad \mathbf{s}\to\mathbf{s} \tag{4.43}$$

This is a canonical transformation since it does not change the commutation relations. The transformation operator is

$$U_P = e^{i\frac{\pi}{2}(\mathbf{p}+i\mathbf{q})\cdot(\mathbf{p}-i\mathbf{q})}. \tag{4.44}$$

The parity transformation has square 1

$$U_P = U_P^{-1} = U_P^\dagger. \tag{4.45}$$

If the parity transformation is an invariance we must have

$$U_P^{-1}HU_P = H \tag{4.46}$$

or

$$[U_P, H] = 0. \tag{4.47}$$

Let us now prove Eq. (4.44) in the one-dimensional case

$$U_P = e^{i\frac{\pi}{2}(p^2+q^2-1)}. \tag{4.48}$$

Apart from a phase this operator coincides with the time evolution operator of a harmonic oscillator of mass 1 and $\omega = 1$ from time $t = 0$ to time $t = \pi$. The Heisenberg equations for

$$q(t) = e^{iHt}q(0)e^{-iHt}, \tag{4.49}$$
$$p(t) = e^{iHt}p(0)e^{-iHt}, \tag{4.50}$$

are

$$\dot{q} = i[H, q], \tag{4.51}$$
$$\dot{p} = i[H, p], \tag{4.52}$$

with $H = (p^2 + q^2)/2$. They have solution

$$q(t) = q\cos t + p\sin t, \tag{4.53}$$
$$p(t) = p\cos t - q\sin t. \tag{4.54}$$

It follows for $t = \pi$

$$q(\pi) = U_P^\dagger q U_P = -q, \tag{4.55}$$
$$p(\pi) = U_P^\dagger p U_P = -p, \tag{4.56}$$

which is what we wanted.

F. Time reversal

The time reversal acts as follows

$$\mathbf{q} \to \mathbf{q} \quad \mathbf{p} \to -\mathbf{p} \quad \mathbf{s} \to -\mathbf{s} \quad t \to -t \tag{4.57}$$

This transformation cannot be realized by a unitary operator because in such case the commutation relations would be preserved. Instead we want, in one dimension,

$$[q, p] = i \to [q, -p] = -i \tag{4.58}$$

If the transformation is antiunitary this is possible:

$$[q', p'] = U_T^\dagger [q, p] U_T = U_T^\dagger i U_T = -i. \tag{4.59}$$

An antilinear operator is defined by

$$T|s_1\rangle = |Ts_1\rangle \quad T|s_2\rangle = |Ts_2\rangle \tag{4.60}$$
$$T(a|s_1\rangle + b|s_2\rangle) = a^* T|s_1\rangle + b^* T|s_2\rangle. \tag{4.61}$$

For a linear operator O

$$\langle a|Ob\rangle = \langle O^\dagger a|b\rangle = \langle b|O^\dagger a\rangle^*, \tag{4.62}$$

and the operator is Hermitian if

$$\langle a|Ob\rangle = \langle Oa|b\rangle. \tag{4.63}$$

For an antilinear operator T

$$\langle a|Tb\rangle = \langle b|T^\dagger a\rangle, \tag{4.64}$$

which is antilinear in $|a\rangle$ and in $|b\rangle$. An antilinear operator is antiunitary if

$$TT^\dagger = T^\dagger T = 1, \tag{4.65}$$

or

$$\langle a|T^\dagger Tb\rangle = \langle Tb|Ta\rangle = \langle a|b\rangle. \tag{4.66}$$

The transformed of O under T

$$O' = T^\dagger OT, \tag{4.67}$$

is still linear and

$$\langle b|T^\dagger OTa\rangle = \langle OTa|Tb\rangle = \langle Ta|O^\dagger Tb\rangle. \tag{4.68}$$

In particular for $O = i$ we find

$$T^\dagger iT = TiT^\dagger = -i. \tag{4.69}$$

The time reversal is realizable with an antiunitary operator:

$$T^\dagger \mathbf{q}T = \mathbf{q} \quad T^\dagger \mathbf{p}T = -\mathbf{p} \quad T^\dagger \mathbf{s}T = -\mathbf{s} \tag{4.70}$$

Moreover, in order to have invariance, we must require

$$T^\dagger HT = H. \tag{4.71}$$

If O is an observable

$$\langle b|OTa\rangle = \langle b|TT^\dagger OTa\rangle = \langle T^\dagger OTa|T^\dagger b\rangle. \tag{4.72}$$

So if $T^\dagger OT = \pm O$ we have

$$\langle b|OTa\rangle = \pm\langle Oa|T^\dagger b\rangle. \tag{4.73}$$

For eigenstates of O, $O|a\rangle = O_a|a\rangle$, we have

$$\langle b|OTa\rangle = \pm O_a\langle a|T^\dagger b\rangle = \pm O_a\langle b|Ta\rangle, \tag{4.74}$$

which means that $|Ta\rangle$ is an eigenstate of O with the transformed eigenvalue.

So for a state $|a\rangle = |\mathbf{p}, s_z\rangle$ we have

$$|Ta\rangle = |-\mathbf{p}, -s_z\rangle, \tag{4.75}$$

modulo a phase.

For a spinless particle with canonical variables $\mathbf{q}$ and $\mathbf{p}$ the time reversal is realized through

$$\langle \mathbf{q}|Ta\rangle = \psi_{Ta}(\mathbf{q}) = \psi_a^*(\mathbf{q}) = \langle \mathbf{q}|a\rangle^*, \tag{4.76}$$

on wave functions in coordinate representation. In fact we have

$$\langle a|T^\dagger \mathbf{p}Tb\rangle = \langle \mathbf{p}Tb|Ta\rangle = \int \psi_b(\mathbf{q})(-i\boldsymbol{\nabla})\psi_a^*(\mathbf{q})\,d\mathbf{q} = -\int \psi_a^*(\mathbf{q})(-i\boldsymbol{\nabla})\psi_b(\mathbf{q})\,d\mathbf{q} = -\langle a|\mathbf{p}b\rangle, \tag{4.77}$$

where we used an integration by parts. Analogously we verify

$$\langle a|T^\dagger \mathbf{q}Tb\rangle = \langle \mathbf{q}Tb|Ta\rangle = \langle a|\mathbf{q}b\rangle. \tag{4.78}$$

The Hamiltonian is an Hermitian function of $\mathbf{q}$ and $\mathbf{p}$. In the coordinate representation, $\mathbf{q}$ is a real variable and $\mathbf{p} = -i\boldsymbol{\nabla}$. The transformation $\mathbf{p} \to -\mathbf{p}$ is equivalent to a complex conjugation. We will have invariance under T if $H(\mathbf{q}, \mathbf{p}) = H(\mathbf{q}, -\mathbf{p})$ or if H is real. A Hamiltonian of the form

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}), \tag{4.79}$$

is invariant under T.

If the particle has spin, it is described by $2s+1$ functions of $\mathbf{q}$

$$\psi(\mathbf{q}) = \begin{pmatrix} \psi_1(\mathbf{q}) \\ \vdots \\ \psi_{2s+1}(\mathbf{q}) \end{pmatrix}. \tag{4.80}$$

The spin is represented by three matrices $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)$ independent from $\mathbf{q}$. We now take

$$\psi_{Ta}(\mathbf{q}) = U\psi_a^*(\mathbf{q}), \tag{4.81}$$

with U an unitary matrix independent from $\mathbf{q}$ and acting on spin space. To have the correct spin transformations we must have

$$\langle a|T^\dagger \mathbf{s} T b\rangle = \langle \mathbf{s} T b|T a\rangle = -\langle a|\mathbf{s} b\rangle, \tag{4.82}$$

or

$$-\int \psi_a^\dagger \mathbf{\Sigma} \psi_b = \int \psi_b^{\mathrm{Tr}} U^\dagger \mathbf{\Sigma} U \psi_a^*, \tag{4.83}$$

which means

$$U^{\mathrm{Tr}} \mathbf{\Sigma}^{\mathrm{Tr}} {U^\dagger}^{\mathrm{Tr}} = -\mathbf{\Sigma}, \tag{4.84}$$

and taking the complex conjugate, since $\mathbf{\Sigma}^\dagger = \mathbf{\Sigma}$, we find

$$U^\dagger \mathbf{\Sigma} U = -\mathbf{\Sigma}^*. \tag{4.85}$$

With the usual choice of phases in the angular momentum representation Σ_1 and Σ_3 are real matrices and Σ_2 is pure imaginary.

For example for spin 1/2 particles

$$\Sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{4.86}$$

Then apart from an unessential phase we find

$$U = e^{i\pi\Sigma_2}, \tag{4.87}$$

a rotation of π around the 2 axis, which changes sign to Σ_1 and Σ_3. In conclusions we have

$$\psi_{Ta} = e^{i\pi\Sigma_2}\psi_a^*. \tag{4.88}$$

V. EINSTEIN RELATIVITY

The invariance under the Galileo group is valid in the limit of small velocities. But, actually, physics is invariant under Lorentz transformations in addition to spatial translations. This invariance is known as Einstein relativity.

The Lorentz group is defined as the group of linear transformations which leaves invariant the quadratic form

$$ds^2 = dt^2 - d\mathbf{x}^2. \tag{5.1}$$

Let $dx = (dx^0, dx^1, dx^2, dx^3) = (dt, d\mathbf{x})$ we can write

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{5.2}$$

where Einstein summation convention is used with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad g^{\mu\nu} g_{\nu\alpha} = \delta^\mu{}_\alpha \tag{5.3}$$

The Lorentz transformations are defined as the linear transformations

$$dx'^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu}, \tag{5.4}$$

such that

$$g_{\mu\nu} dx'^{\mu} dx'^{\nu} = g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} dx^{\alpha} dx^{\beta}. \tag{5.5}$$

Due to the arbitrariness of dx^{μ} we have

$$g_{\mu\nu} = g_{\alpha\beta} \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu}, \tag{5.6}$$

or

$$\mathbf{g} = \mathbf{\Lambda}^{\mathrm{Tr}} \mathbf{g} \mathbf{\Lambda}, \tag{5.7}$$

which defines the Lorentz group. Taking the 00 component in Eq. (5.6)

$$1 = g_{\alpha\beta} \Lambda^{\alpha}{}_{0} \Lambda^{\beta}{}_{0} = (\Lambda^{0}{}_{0})^2 - \sum_i (\Lambda^{i}{}_{0})^2, \tag{5.8}$$

or

$$(\Lambda^{0}{}_{0})^2 \geq 1, \tag{5.9}$$

or

$$\Lambda^{0}{}_{0} \geq 1 \quad \text{or} \quad \Lambda^{0}{}_{0} \leq -1. \tag{5.10}$$

Taking the determinant in Eq. (5.6) follows

$$(\det \mathbf{\Lambda})^2 = 1, \tag{5.11}$$

or

$$\det \mathbf{\Lambda} = \pm 1. \tag{5.12}$$

The transformations obtained continuously from the identity have $\Lambda^{0}{}_{0} \geq 1$ and $\det \mathbf{\Lambda} = 1$ and constitute the *proper* Lorentz group. The transformations with $\Lambda^{0}{}_{0} \geq 1$ and $\det \mathbf{\Lambda} = -1$ can be written as the product of the parity $P : \mathbf{x} \to -\mathbf{x}$ times a proper transformation. The ones with $\Lambda^{0}{}_{0} \leq -1$ and $\det \mathbf{\Lambda} = 1$ as a product of the time reversal $T : x^0 \to -x^0$ times the proper transformations. The ones with $\Lambda^{0}{}_{0} \leq 1$ and $\det \mathbf{\Lambda} = -1$ as PT times a proper transformation.

An infinitesimal proper transformation

$$\Lambda^{\mu}{}_{\mu'} = \delta^{\mu}{}_{\mu'} + \Omega^{\mu}{}_{\mu'}, \tag{5.13}$$

must satisfy Eq. (5.6). So

$$g_{\mu'\nu'} = g_{\mu'\nu'} + \Omega^{\mu}{}_{\mu'} g_{\mu\nu'} + \Omega^{\nu}{}_{\nu'} g_{\nu\mu'} + \mathcal{O}(\mathbf{\Omega}^2). \tag{5.14}$$

Let

$$\Omega_{\mu\nu} = g_{\mu\alpha} \Omega^{\alpha}{}_{\nu}, \tag{5.15}$$

then we must have

$$\Omega_{\mu\nu} = -\Omega_{\nu\mu}. \tag{5.16}$$

The group has 6 parameters as the number of components of an antisymmetric 4×4 matrix. The most general 4×4 antisymmetric matrix can be written as

$$\Omega_{\mu\nu} = \frac{1}{2} \sum_{\rho\sigma} \omega_{(\rho\sigma)} M^{(\rho\sigma)}_{\mu\nu}, \tag{5.17}$$

$$M^{(\rho\sigma)}_{\mu\nu} = \delta^{\rho}{}_{\mu} \delta^{\sigma}{}_{\nu} - \delta^{\rho}{}_{\nu} \delta^{\sigma}{}_{\mu} = -M^{(\sigma\rho)}_{\mu\nu}. \tag{5.18}$$

We write

$$\left(M^{(\rho\sigma)}\right)^{\mu}_{\ \ \nu} = g^{\mu\alpha} M^{(\rho\sigma)}_{\alpha\nu}, \tag{5.19}$$

so

$$\Omega^{\mu}_{\ \nu} = g^{\mu\alpha}\Omega_{\alpha\nu} = g^{\mu\alpha}\frac{1}{2}\omega_{(\rho\sigma)} M^{(\rho\sigma)}_{\mu\nu} = \frac{1}{2}\omega_{(\rho\sigma)}\left(M^{(\rho\sigma)}\right)^{\mu}_{\ \ \nu}. \tag{5.20}$$

The matrices $M^{(\mu\nu)}$ satisfy the following algebra

$$[M^{(\alpha\beta)}, M^{(\mu\nu)}] = -\left(g^{\alpha\mu}M^{(\beta\nu)} + g^{\beta\nu}M^{(\alpha\mu)} - g^{\beta\mu}M^{(\alpha\nu)} - g^{\alpha\nu}M^{(\beta\mu)}\right). \tag{5.21}$$

We can then introduce

$$J^{(\mu\nu)} \equiv -iM^{(\mu\nu)}, \tag{5.22}$$

and

$$J^i = -\frac{1}{2}\epsilon_{0ijk}J^{(jk)}, \tag{5.23}$$

$$K^i = J^{(0i)}, \tag{5.24}$$

where $\epsilon_{\mu_0\mu_1\mu_2\mu_3}$ is the Levi-Civita symbol with $\epsilon_{0123} = 1$ [1]. Then Eq. (5.21) is rewritten as

$$[J^i, J^j] = i\epsilon_{ijk}J^k, \tag{5.28}$$

$$[J^i, K^j] = i\epsilon_{ijk}K^k, \tag{5.29}$$

$$[K^i, K^j] = -i\epsilon_{ijk}J^k. \tag{5.30}$$

The generators J^i are the rotations generators, which constitute a subgroup of the Lorentz transformations. The K^i are the generators of the velocity ($\mathbf{v}$) transformations and are vectors, as follows from their commutation relations with the J^i. The infinitesimal transformations are then

$$\mathbf{\Lambda} = 1 + i(\boldsymbol{\theta}\cdot\mathbf{J} - \boldsymbol{\alpha}\cdot\mathbf{K}). \tag{5.31}$$

The finite ones are

$$\mathbf{\Lambda} = e^{\frac{i}{2}\sum_{\alpha\beta} J^{(\alpha\beta)}\omega_{(\alpha\beta)}} = e^{i(\boldsymbol{\theta}\cdot\mathbf{J} - \boldsymbol{\alpha}\cdot\mathbf{K})} \quad \mathbf{v} = (\tanh\alpha_1, \tanh\alpha_2, \tanh\alpha_3), \tag{5.32}$$

where $\boldsymbol{\theta}$ is the rotation angle vector and $\boldsymbol{\alpha}$ is the rapidity vector.

Under the Lorentz group the generators of the translations p_μ must transform as four-vectors

$$[J^{(\mu\nu)}, p^\alpha] = i(g^{\mu\alpha}p^\nu - g^{\alpha\nu}p^\mu), \tag{5.33}$$

or

$$[\mathbf{J}, p^0] = -\delta p^0 = -\mathbf{J}p^0 = 0, \tag{5.34}$$

which expresses the conservation of angular momentum, and

$$[J^i, p^j] = -\delta p^j = -J^i p^j = i\epsilon_{ijk}p^k, \tag{5.35}$$

---

[1] For any antisymmetric tensor $F^{\mu\nu}$ it is possible to use a decomposition of the following kind: $F^{\mu\nu} = (\mathbf{P}, \mathbf{A})$ with

$$A^1 = -F^{23} \quad A^2 = -F^{31} \quad A^3 = -F^{12} \tag{5.25}$$

$$P^1 = F^{01} \quad P^2 = F^{02} \quad P^3 = F^{03} \tag{5.26}$$

For the product of two tensors of this kind we have

$$\frac{1}{2}F^{(1)}_{\mu\nu}F^{(2)\mu\nu} = \mathbf{A}^{(1)}\cdot\mathbf{A}^{(2)} - \mathbf{P}^{(1)}\cdot\mathbf{P}^{(2)}. \tag{5.27}$$

which tells us that $\mathbf{p}$ is a vector. On the momenta the generators of the velocity transformations act as follows

$$[K^i, p^0] = -\delta p^0 = -K^i p^0 = ig^{00} p^i, \tag{5.36}$$
$$[K^i, p^j] = -\delta p^j = -K^i p^j = -ig^{ij} p^0. \tag{5.37}$$

The invariance under translations is written as

$$[p^\mu, p^\nu] = 0. \tag{5.38}$$

The commutation relations between the generators are then

$$[p^\mu, p^\nu] = 0, \tag{5.39}$$
$$[J^{(\mu\nu)}, p^\alpha] = i(g^{\mu\alpha} p^\nu - g^{\alpha\nu} p^\mu), \tag{5.40}$$
$$[J^{(\alpha\beta)}, J^{(\mu\nu)}] = i\left(g^{\alpha\mu} J^{(\beta\nu)} + g^{\beta\nu} J^{(\alpha\mu)} - g^{\beta\mu} J^{(\alpha\nu)} - g^{\alpha\nu} J^{(\beta\mu)}\right). \tag{5.41}$$

They define the Lie algebra of a 10 parameters group known as the Poincaré group.

The Poincaré group is defined by the transformation laws

$$(\Lambda, a) : x \to x' = \mathbf{\Lambda} x - a, \tag{5.42}$$

where a is a translation and Λ is a Lorentz transformation. We immediately find the multiplication properties of the group as

$$(\Lambda_1, a)(\Lambda_2, b) = (\Lambda_1 \Lambda_2, -\Lambda_1 b - a), \tag{5.43}$$

from which immediately follows that the translations are an abelian invariant subgroup. In fact applying repetitively Eq. (5.43) we find that the transformed by similitude of a translation $(1, a)$,

$$(\Lambda, c)(1, a)(\Lambda^{-1}, -c) = (1, \Lambda(c - a) - c), \tag{5.44}$$

is still a translation.

By Wigner theorem the states of a physical system are the basis of a unitary representation of the Poincaré group. An elementary system will be described by an irreducible representation of the Poincaré group.

We note that

$$\mathbf{J}_\pm = \frac{\mathbf{J} \pm i\mathbf{K}}{2}, \tag{5.45}$$

obey the following commutation relations

$$[J^i_+, J^j_+] = i\epsilon_{ijk} J^k_+, \tag{5.46}$$
$$[J^i_-, J^j_-] = i\epsilon_{ijk} J^k_-, \tag{5.47}$$
$$[J^i_+, J^j_-] = 0. \tag{5.48}$$

So the generators of $\mathbf{J}_+$ and $\mathbf{J}_-$ obey to the algebra $SU(2) \otimes SU(2)$. Let us show now that an irreducible representation of the Poincaré group, i.e. an elementary particle, is determined by the mass and the spin.

An irreducible representation is characterized by the value of the *invariants*, i.e. of the operators built with the generators of the group that commute with all the group generators. We then define

$$\Gamma_\mu = \frac{1}{2} \epsilon_{\mu\alpha\beta\sigma} J^{(\alpha\beta)} p^\sigma, \tag{5.49}$$
$$\Gamma_\mu p^\mu = 0, \tag{5.50}$$
$$g^\mu = J^{(\mu\nu)} p_\nu, \tag{5.51}$$
$$g^\mu p_\mu = 0. \tag{5.52}$$

One can prove (Shirokov, 1958a,b, 1959) [2] that

$$p^2 J^{(\mu\nu)} = g^\mu p^\nu - g^\nu p^\mu - \epsilon^{\sigma\mu\nu\lambda} \Gamma_\sigma p_\lambda. \tag{5.54}$$

[2] One can use the identity

$$\epsilon^{\sigma\mu\nu\lambda} \epsilon_{\sigma\alpha\beta\rho} = \det \begin{pmatrix} \delta^\mu_\alpha & \delta^\mu_\beta & \delta^\mu_\rho \\ \delta^\nu_\alpha & \delta^\nu_\beta & \delta^\nu_\rho \\ \delta^\lambda_\alpha & \delta^\lambda_\beta & \delta^\lambda_\rho \end{pmatrix}, \tag{5.53}$$

and the definition of Γ_σ to calculate the product $\epsilon^{\sigma\mu\nu\lambda} \Gamma_\sigma p_\lambda$.

This tells us that $J^{(\mu\nu)}$ can be expressed in terms of p_μ, Γ_μ, and g_μ if $p^2 = p_\mu p^\mu \neq 0$.

Moreover we have

$$[\Gamma_\mu, \Gamma_\nu] = i\epsilon_{\rho\mu\nu\lambda}\Gamma^\rho p^\lambda, \quad (5.55)$$

$$[g_\mu, \Gamma_\sigma] = -i\Gamma_\mu p_\sigma, \quad (5.56)$$

$$[g_\mu, p_\nu] = i(g_{\mu\nu}p^2 - p_\mu p_\nu), \quad (5.57)$$

$$[g_\mu, g_\nu] = -i(g^\mu p^\nu - g^\nu p^\mu - \epsilon^{\sigma\mu\nu\lambda}\Gamma_\sigma p_\lambda) \quad (5.58)$$

$$[p_\mu, \Gamma_\sigma] = 0. \quad (5.59)$$

An invariant should be constructed with the vectors p_μ, Γ_μ, and g_μ. Recalling that $g_\mu p^\mu = 0$ and $\Gamma_\mu p^\mu = 0$ the only independent invariants under the Lorentz group are

$$p^2, \Gamma^2, g^2, \Gamma_\mu g^\mu. \quad (5.60)$$

But g^2 and $\Gamma_\mu g^\mu$ do not commute with translations. Then the representation is determined by p^2, Γ^2, and by the sign of p^0, which is also invariant under the proper Lorentz group and commutes with translations, if $p^2 \geq 0$.

The physical interpretation of the two invariants is obvious:

i. For the invariant p^2 we have 4 cases

$$p^2 > 0, \quad (5.61)$$

$$p^2 = 0 \quad p \neq 0, \quad (5.62)$$

$$p^2 = 0 \quad p = 0, \quad (5.63)$$

$$p^2 < 0. \quad (5.64)$$

Since $p^2 = m^2$ we will be interested only in the first two cases. In these two cases, for the representations of the proper group (${\Lambda^0}_0 \geq 0$ and $\det\Lambda = 1$) we will have another invariant, namely the sign of p^0.

ii. The invariant Γ^2 can be calculated in the reference frame where $\mathbf{p} = 0$. In such a frame

$$\Gamma = (\Gamma^0, \Gamma^1, \Gamma^2, \Gamma^3) = (\Gamma^0, \mathbf{\Gamma}) = (0, m\mathbf{J}) \quad \Gamma^2 = -m^2 J(J+1). \quad (5.65)$$

The modulus of $\mathbf{J}$ in the rest frame is by definition the particle spin, so $\Gamma^2 = -m^2 s(s+1)$

Then the representation is determined by the mass m and by the spin s, exactly as in the nonrelativistic happens for the Galileo group.

A. The irreducible unitary representation of the Poincaré group

We want now to explicitly construct the irreducible unitary representations of the Poincaré group.

1. Massive particles

We can build a base of the Hilbert space which diagonalizes simultaneously the components p_μ of the four-momentum, which commute among themselves, and other observables which we will denote by now with σ. The vector of the base will have the form $|\mathbf{p}, \sigma\rangle$ with

$$p_i|\mathbf{p}, \sigma\rangle = p_i|\mathbf{p}, \sigma\rangle \quad p_0|\mathbf{p}, \sigma\rangle = \mathrm{sgn}(p_0)p_0|\mathbf{p}, \sigma\rangle, \quad (5.66)$$

with $p_0 = p^0 \equiv \sqrt{\mathbf{p}^2 + m^2}$ and $\mathbf{p} = (p^1, p^2, p^3) = (-p_1, -p_2, -p_3)$. We will call $U(\Lambda)$ the unitary operators which represents the generic Lorentz transformation $\mathbf{\Lambda}$. We will have

$$U(\Lambda)|\mathbf{p}, \sigma\rangle = \sum_{\sigma'} \mathcal{R}(\Lambda, \mathbf{p})_{\sigma\sigma'}|\Lambda\mathbf{p}, \sigma'\rangle. \quad (5.67)$$

In fact, using the group algebra we have

$$U^\dagger(\Lambda)p_\mu U(\Lambda) = {\Lambda^\nu}_\mu p_\nu, \quad (5.68)$$

then

$$p_\mu U(\Lambda)|\mathbf{p},\sigma\rangle = U(\Lambda)U^\dagger(\Lambda)p_\mu U(\Lambda)|\mathbf{p},\sigma\rangle = \Lambda^\nu{}_\mu p_\nu U(\Lambda)|\mathbf{p},\sigma\rangle. \quad (5.69)$$

So $U(\Lambda)|\mathbf{p},\sigma\rangle$ belongs to the eigenvalue $(\mathbf{\Lambda}p)_\mu$ of the four-momentum. And this proves Eq. (5.67). The Lorentz invariant measure, for momentum $p = (p^0, p^1, p^2, p^3) = (p^0, \mathbf{p})$, is

$$d\Omega_{\mathbf{p}} = \frac{d^4p}{(2\pi)^3}\delta(\sqrt{p^2} - m)\theta(p_0) = \frac{d^3\mathbf{p}}{(2\pi)^3 2p_0}. \quad (5.70)$$

One can easily verify that with the invariant normalization

$$\langle\mathbf{p}',\sigma'|\mathbf{p},\sigma\rangle = (2\pi)^3 2p_0\delta(\mathbf{p}-\mathbf{p}')\delta_{\sigma\sigma'}, \quad (5.71)$$

the matrix $\mathcal{R}(\Lambda,\mathbf{p})_{\sigma\sigma'}$ in Eq. (5.67) is unitary due to the unitarity of $U(\Lambda)$.

The operator Γ_μ commutes with all the components of p_μ. Then when applied to the state $|\mathbf{p},\sigma\rangle$ it can only mix it with states of the same $\mathbf{p}$.

Let us start by considering the case $p^2 = m^2 > 0$ with $\mathbf{p} = \mathbf{0}$, $|\mathbf{0},\sigma\rangle$, for which

$$\mathbf{p}|\mathbf{0},\sigma\rangle = \mathbf{0} \quad p_0|\mathbf{0},\sigma\rangle = \mathrm{sgn}(p_0)m|\mathbf{0},\sigma\rangle. \quad (5.72)$$

On this subspace $\Gamma_\mu = \frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}J^{(\alpha\beta)}p^\gamma$ can be easily calculated

$$\Gamma_0 = 0 \quad \mathbf{\Gamma} = m\mathbf{J} \equiv m\mathbf{s}. \quad (5.73)$$

The angular momentum of the rest frame is called spin by definition. The dimension of the subspace is $2s+1$.

For the variable σ we can take the eigenvalue of one of the spin component, i.e. s_3.

If $U(\Lambda_{\mathbf{p}})$ is a Lorentz transformation which brings the momentum from $\mathbf{0}$ to a certain value $\mathbf{p}$, since Γ^μ is a four-vector, we will have

$$U^\dagger(\Lambda_{\mathbf{p}})\Gamma^\mu U(\Lambda_{\mathbf{p}}) = (\Lambda_{\mathbf{p}})^\mu{}_\nu\Gamma^\nu. \quad (5.74)$$

If we call $\bar\Gamma^\mu_{\sigma'\sigma}$ the representative of the Γ^μ on the subspace $|\mathbf{0},\sigma\rangle$ we will have

$$\begin{aligned}\Gamma^\mu U(\Lambda_{\mathbf{p}})|\mathbf{0},\sigma\rangle &= U(\Lambda_{\mathbf{p}})U^\dagger(\Lambda_{\mathbf{p}})\Gamma^\mu U(\Lambda_{\mathbf{p}})|\mathbf{0},\sigma\rangle \\ &= U(\Lambda_{\mathbf{p}})(\Lambda_{\mathbf{p}})^\mu{}_\nu\Gamma^\nu|\mathbf{0},\sigma\rangle \\ &= (\Lambda_{\mathbf{p}})^\mu{}_\nu\bar\Gamma^\nu_{\sigma'\sigma}U(\Lambda_{\mathbf{p}})|\mathbf{0},\sigma\rangle. \quad (5.75)\end{aligned}$$

Then $(\Lambda_{\mathbf{p}})^\mu{}_\nu\bar\Gamma^\nu_{\sigma'\sigma}$ is the representative of Γ^μ on the subspace $|\mathbf{p},\sigma\rangle$, in the representation in which the base vectors are $|\mathbf{p},\sigma\rangle = U(\Lambda_{\mathbf{p}})|\mathbf{0},\sigma\rangle$.

The Lorentz transformation $U(\Lambda_{\mathbf{p}})$ which brings the momentum from $\mathbf{0}$ to $\mathbf{p}$ is not univocally defined: it is indetermined on the right by a transformation of the small group [3] of the initial momentum $\mathbf{0}$ and on the left by a transformation of the small group of the final momentum $\mathbf{p}$.

For each choice of these transformations we will have a choice of the base vectors $U(\Lambda_{\mathbf{p}})|\mathbf{0},\sigma\rangle$ and of the representative of Γ^μ. We will adopt, in the following, a standard choice for $U(\Lambda_{\mathbf{p}})$. Namely a simple velocity transformation $e^{-i\boldsymbol{\alpha}\cdot\mathbf{K}}$ in the $\mathbf{p}$ direction, which sends the momentum from $\mathbf{0}$ to $\mathbf{p}$. The base vectors are then

$$|\mathbf{p},\sigma\rangle = U(\Lambda_{\mathbf{p}})|\mathbf{0},\sigma\rangle = e^{-i\boldsymbol{\alpha}\cdot\mathbf{K}}|\mathbf{0},\sigma\rangle, \quad (5.76)$$

and

$$\Gamma^\mu(\mathbf{p}) = (\Lambda_{\mathbf{p}})^\mu{}_\nu\Gamma^\nu(\mathbf{0}) = \left(\mathbf{p}\cdot\mathbf{s}, m\mathbf{s} + \frac{(\mathbf{p}\cdot\mathbf{s})\mathbf{p}}{p^0+m}\right). \quad (5.77)$$

[3] The small group of $\mathbf{p}$ is the subgroup of the transformations which leaves $\mathbf{p}$ unchanged.

This can be proved as follows. We can write for a general velocity transformation

$$\mathbf{\Lambda_p} = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta}^{\mathrm{Tr}} \\ -\gamma\boldsymbol{\beta} & \mathbf{1} + (\gamma-1)\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{Tr}}/\beta^2 \end{pmatrix} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \tag{5.78}$$

with

$$-\gamma\boldsymbol{\beta} = \frac{\mathbf{p}}{m} \implies \gamma = \frac{p_0}{m} \quad \text{and} \quad (\gamma-1)/\beta^2 = \frac{p_0^2}{m(p_0+m)} \tag{5.79}$$

The transformation we are looking for is then

$$(\Lambda_{\mathbf{p}})^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \frac{1}{p_0+m}\left[m\delta^{\mu}{}_0\delta^0{}_{\nu} + \delta^{\mu}{}_0 p_{\nu} + \frac{p^{\mu}p_{\nu}}{m} - \delta^0{}_{\nu}p^{\mu}\left(1+2\frac{p_0}{m}\right)\right], \tag{5.80}$$

from which we immediately find Eq. (5.77).

To complete the construction of the representation we could now look for the representative of g_μ defined in (5.51), using the commutation relations (5.55)-(5.59), and construct the representative of the generic $J^{(\mu\nu)}$ using Eq. (5.54). Alternatively we may proceed as follows:

a) Let us first consider the rotations. If Λ is a rotation $R_{\boldsymbol{\theta}}$

$$U(R_{\boldsymbol{\theta}})|\mathbf{p},\sigma\rangle = U(R_{\boldsymbol{\theta}})U(\Lambda_{\mathbf{p}})U^{\dagger}(R_{\boldsymbol{\theta}})U(R_{\boldsymbol{\theta}})|\mathbf{0},\sigma\rangle. \tag{5.81}$$

We know that

$$U(\Lambda_{\mathbf{p}}) = e^{-i\boldsymbol{\alpha}\cdot\mathbf{K}}, \tag{5.82}$$

and since

$$U^{\dagger}(R_{\boldsymbol{\theta}})\mathbf{K}U(R_{\boldsymbol{\theta}}) = R_{\boldsymbol{\theta}}\mathbf{K}, \tag{5.83}$$

we have

$$U(R_{\boldsymbol{\theta}})U(\Lambda_{\mathbf{p}})U^{\dagger}(R_{\boldsymbol{\theta}}) = e^{-i(R_{\boldsymbol{\theta}}\boldsymbol{\alpha})\cdot\mathbf{K}}, \tag{5.84}$$

and

$$U(R_{\boldsymbol{\theta}})|\mathbf{p},\sigma\rangle = \left(e^{i\boldsymbol{\theta}\cdot\mathbf{s}}\right)_{\sigma'\sigma}|R_{\boldsymbol{\theta}}\mathbf{p},\sigma'\rangle. \tag{5.85}$$

b) For a Lorentz transformation sending $\mathbf{p}$ into $\mathbf{p}'$

$$U(\Lambda)|\mathbf{p},\sigma\rangle = U(\Lambda_{\mathbf{p}'})U^{\dagger}(\Lambda_{\mathbf{p}'})U(\Lambda)U(\Lambda_{\mathbf{p}})|\mathbf{0},\sigma\rangle. \tag{5.86}$$

The matrix $U^{\dagger}(\Lambda_{\mathbf{p}'})U(\Lambda)U(\Lambda_{\mathbf{p}})$ belongs to the small group of $\mathbf{p}=\mathbf{0}$, i.e. it is a rotation $R(\Lambda,\mathbf{p})$ in the subspace $|\mathbf{0},\sigma\rangle$. To determine it we just need to calculate

$$\mathbf{\Lambda}_{\mathbf{p}'}^{-1}\mathbf{\Lambda}\mathbf{\Lambda}_{\mathbf{p}}, \tag{5.87}$$

using the formula (5.80) and the explicit one (5.78). If we call $\mathcal{R}(\Lambda,\mathbf{p})_{\sigma'\sigma}$ the representative of such a rotation in the space $|\mathbf{0},\sigma\rangle$ we will have

$$U(\Lambda)|\mathbf{p},\sigma\rangle = \mathcal{R}(\Lambda,\mathbf{p})_{\sigma'\sigma}|\Lambda\mathbf{p},\sigma\rangle. \tag{5.88}$$

Explicitly, if Λ is a velocity transformation with velocity β in the direction $\hat{\mathbf{n}}$, we find

$$\begin{aligned}(\mathbf{\Lambda}_{\mathbf{p}'}^{-1}\mathbf{\Lambda}\mathbf{\Lambda}_{\mathbf{p}},)^{\mu}{}_{\nu} &= \begin{pmatrix} \bar{\mathcal{R}}^0{}_0 & \bar{\mathcal{R}}^0{}_j \\ \bar{\mathcal{R}}^i{}_0 & \bar{\mathcal{R}}^i{}_j \end{pmatrix} \\ &= \begin{pmatrix} \frac{p_0'}{m} & -\frac{p_j'}{m} \\ -\frac{p_i'}{m} & \delta_{ik} + \frac{p_i'p_k'}{m(p_0'+m)} \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta n_l \\ -\gamma\beta n_k & \delta_{kl} + (\gamma-1)n_k n_l \end{pmatrix} \begin{pmatrix} \frac{p_0}{m} & \frac{p_j}{m} \\ \frac{p_l}{m} & \delta_{lj} + \frac{p_l p_j}{m(p_0+m)} \end{pmatrix}\end{aligned} \tag{5.89}$$

To first order in β

$$\begin{aligned} p_0' &= \gamma p_o - \gamma\beta\hat{\mathbf{n}}\cdot\mathbf{p} = p_0 - \boldsymbol{\beta}\cdot\mathbf{p} + \mathcal{O}(\beta^2), \qquad (5.90)\\ \mathbf{p}' &= -\gamma\beta\hat{\mathbf{n}}p_0 + \mathbf{p} + (\gamma-1)\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p}) = \mathbf{p} - \boldsymbol{\beta}p_0 + \mathcal{O}(\beta^2), \qquad (5.91)\end{aligned}$$

and

$$\bar{\mathcal{R}}^i{}_j = \delta^i{}_j + \frac{\beta}{p_0+m}(n^i p_j - p^i n_j) + \mathcal{O}(\beta^2). \tag{5.92}$$

Recalling that in the vector representation $(J^i)_{jk} = i\epsilon_{jik}$ and using $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ we finally find

$$\mathcal{R} \approx 1 - i\frac{\boldsymbol{\beta}\wedge\mathbf{p}}{p_0+m}\cdot\mathbf{J}. \tag{5.93}$$

The finite transformation $\mathcal{R}(\Lambda,\mathbf{p})$ is then of the following form

$$\mathcal{R}(\Lambda,\mathbf{p}) = e^{-i\frac{\boldsymbol{\beta}\wedge\mathbf{p}}{p_0+m}\cdot\mathbf{J}}. \tag{5.94}$$

This rotation is called the Wigner rotation.

The transformations on the wave functions can be determined from the one on the states. The generic $|\phi\rangle$ is written as

$$|\phi\rangle = \int d\Omega_{\mathbf{p}}\,\varphi_\sigma(\mathbf{p})|\mathbf{p},\sigma\rangle, \tag{5.95}$$

and the scalar product

$$\langle\phi'|\phi\rangle = \int d\Omega_{\mathbf{p}}\,{\varphi'}^*(\mathbf{p})\varphi(\mathbf{p}). \tag{5.96}$$

Under a transformation $U(\Lambda,a)$

$$|\Lambda\phi\rangle = U(\Lambda,a)|\phi\rangle = \int d\Omega_{\mathbf{p}}\,e^{-ipa}\varphi_\sigma(\mathbf{p})\mathcal{R}(\Lambda,\mathbf{p})_{\sigma'\sigma}|\Lambda\mathbf{p},\sigma'\rangle. \tag{5.97}$$

Changing variables from $\mathbf{p}$ to $\Lambda^{-1}\mathbf{p}$ and using the fact that the measure is invariant we find

$$|\Lambda\phi\rangle = \int d\Omega_{\mathbf{p}}\,e^{-i(\Lambda^{-1}p)a}\mathcal{R}_{\sigma'\sigma}\varphi_\sigma(\Lambda^{-1}\mathbf{p})|\mathbf{p},\sigma'\rangle, \tag{5.98}$$

or

$$(\Lambda\varphi)_\sigma(\mathbf{p}) = \mathcal{R}_{\sigma\sigma'}\varphi_{\sigma'}(\Lambda^{-1}\mathbf{p}). \tag{5.99}$$

The matrix $\mathcal{R}$ is given by Eq. (5.85) for the rotations and by Eq. (5.89) for the velocity transformations and is unitary respect to the metric $\langle\phi'|\phi\rangle$.

For an infinitesimal transformation Eq. (5.99) gives the form of the generators:

a) For an infinitesimal rotation of an angle $\boldsymbol{\theta}$

$$\Lambda^{-1}\mathbf{p} \approx \mathbf{p} + \boldsymbol{\theta}\wedge\mathbf{p} \quad \mathcal{R} \approx 1 + i\boldsymbol{\theta}\cdot\mathbf{s}, \tag{5.100}$$

so

$$\delta\varphi_\sigma \equiv (\Lambda\varphi_\sigma) - \varphi_\sigma \approx i\boldsymbol{\theta}\cdot\mathbf{s}_{\sigma\sigma'}\varphi_{\sigma'} + (\boldsymbol{\theta}\wedge\mathbf{p})\frac{\partial}{\partial\mathbf{p}}\varphi_\sigma. \tag{5.101}$$

The generator is defined by $\delta\varphi = i(\boldsymbol{\theta}\cdot\mathbf{J})\varphi$ so

$$\mathbf{J} = \mathbf{s} - i\mathbf{p}\wedge\frac{\partial}{\partial\mathbf{p}}. \tag{5.102}$$

b) For a velocity transformation

$$\Lambda^{-1}\mathbf{p} \approx \mathbf{p} + \beta p_0 \quad \mathcal{R} \approx 1 - i\frac{\boldsymbol{\beta}\wedge\mathbf{p}}{p_0+m}\cdot\mathbf{s}, \tag{5.103}$$

so

$$\delta\varphi_\sigma \approx -i(\boldsymbol{\beta}\cdot\mathbf{K})\varphi_\sigma = -i\frac{\boldsymbol{\beta}\wedge\mathbf{p}}{p_0+m}\cdot\mathbf{s}_{\sigma\sigma'}\varphi_{\sigma'} + \boldsymbol{\beta}\cdot p_0\frac{\partial}{\partial\mathbf{p}}\varphi_\sigma, \tag{5.104}$$

or

$$\mathbf{K} = \frac{\mathbf{p}\wedge\mathbf{s}}{p_0+m} + ip_0\frac{\partial}{\partial\mathbf{p}}. \tag{5.105}$$

One can verify that the generators $\mathbf{J}$ and $\mathbf{K}$ satisfy the algebra of the group. This completes the construction of the representation of the group on the Hilbert space of the multiplets of functions $\varphi(\mathbf{p})$ with the metric of Eq. (5.96).

2. The Elicity

We just saw that states can be taken as simultaneous eigenstates of p^2 and Γ^2 and accordingly labeled as $|m, s, \ldots\rangle$, with

$$p^2|m,s,\ldots\rangle = m^2|m,s,\ldots\rangle, \tag{5.106}$$

$$\Gamma^2|m,s,\ldots\rangle = -ms(s+1)|m,s,\ldots\rangle. \tag{5.107}$$

What are the additional quantum numbers we can use to label the states? They must be eigenvalues of operators which commute with each other. So we are free to consider states of definite four-momentum p_μ. Since the mass is already fixed, it is only necessary to specify in addition the three-momentum $\mathbf{p}$, the energy being determined by $p^0 = \sqrt{m^2+\mathbf{p}^2}$. We cannot simultaneously give definite value for the third component of the angular momentum operator J^3 because $\mathbf{J}$ and $\mathbf{p}$ do not commute. However there is an angular momentum operator which commutes with $\mathbf{p}$, namely the *elicity*. This operator is the component of the spin along the direction of the momentum, $\mathbf{J}\cdot\mathbf{p}/|\mathbf{p}|$, and its eigenvalues are labeled a. Thus the complete specification of the momentum eigenstates of a massive particle is $|m, s; \mathbf{p}, a\rangle$ with

$$p_\mu|m,s;\mathbf{p},a\rangle = p_\mu|m,s;\mathbf{p},a\rangle, \tag{5.108}$$

$$\frac{\mathbf{J}\cdot\mathbf{p}}{|\mathbf{p}|}|m,s;\mathbf{p},a\rangle = a|m,s;\mathbf{p},a\rangle. \tag{5.109}$$

3. Massless particles

Let us consider the base $|\mathbf{p},\sigma\rangle$. For $p^2 = 0$ it does not exist a rest frame. We will take as the standard state the state with

$$p^\mu = (\bar{p}, 0, 0, \bar{p}), \tag{5.110}$$

with $\bar{p}$ chosen arbitrarily. We will call $|\bar{\mathbf{p}},\sigma\rangle$ the corresponding subspace. We will assume $\mathrm{sgn}(p^0) = 1$. The discussion for $\mathrm{sgn}(p^0) = -1$ is analogous.

On the states $|\bar{\mathbf{p}},\sigma\rangle$, Γ^μ acts mixing them, since it commutes with p^μ. The condition $\Gamma^\mu p_\mu = 0$ gives

$$\bar{p}(\Gamma^0 - \Gamma^3) = 0. \tag{5.111}$$

We will define

$$\Gamma^0 = \bar{p}\tilde{\Gamma} \quad \Gamma^\pm = \Gamma^1 \pm \Gamma^2. \tag{5.112}$$

The the commutation rule

$$[\Gamma_\mu, \Gamma_\nu] = i\epsilon_{\mu\nu\rho\lambda}\Gamma^\rho p^\lambda, \tag{5.113}$$

gives

$$\begin{aligned}
[\Gamma^{\pm}, \tilde{\Gamma}] &= \mp\Gamma^{\pm}, &(5.114)\\
[\Gamma^{+}, \Gamma^{-}] &= 0, &(5.115)\\
[\Gamma^{2}, \tilde{\Gamma}] &= 0, &(5.116)\\
[\Gamma^{2}, \Gamma^{\pm}] &= 0. &(5.117)
\end{aligned}$$

Moreover

$$\Gamma^2 = \Gamma^+\Gamma^-, \tag{5.118}$$

since $\Gamma_0^2 - \Gamma_3^2 = 0$.

In a unitary representation $\Gamma^+ = (\Gamma^-)^\dagger$. If we diagonalize $\tilde{\Gamma}$,

$$\tilde{\Gamma}|\bar{\mathbf{p}}, a\rangle = a|\bar{\mathbf{p}}, a\rangle, \tag{5.119}$$

from Eq. (5.114) the operators Γ^+ and Γ^- are the operators of highering and lowering of a respectively. Their representative is then

$$\begin{aligned}
(\Gamma^{+})_{mn} &= b_n\delta_{m,n+1}, &(5.120)\\
(\Gamma^{-})_{mn} &= b_m^*\delta_{n,m+1}. &(5.121)\\
& &(5.122)
\end{aligned}$$

Then Eq. (5.118) imposes $\Gamma^2 = |b_n|^2 = \alpha^2$, independent from n. If $\alpha \neq 0$ the Γ^μ representation is infinite dimensional. In order to have a finite number of states of fixed spin and momentum it must be $\alpha = 0$. This implies $\Gamma^+ = \Gamma^- = 0$ and, by Eq. (5.118), $\Gamma^2 = 0$. So we can say that

$$\Gamma^\mu = \tilde{\Gamma}p^\mu. \tag{5.123}$$

The physical significance of $\tilde{\Gamma}$ can be obtained from the definition (5.49) of Γ^μ

$$\tilde{\Gamma} = \frac{\mathbf{J}\cdot\mathbf{p}}{|\mathbf{p}|}. \tag{5.124}$$

$\tilde{\Gamma}$ is the projection of the spin on the direction of motion, i.e. the elicity.

From Eq. (5.123) follows that $\tilde{\Gamma}$ is an invariant. For a massless particle the elicity is a Poincaré invariant. The representation is one dimensional. The elicity is a pseudoscalar: A representation with a fixed elicity defines a system which is not invariant under parity because the transformed state has opposite elicity and is not a possible state. The invariance under parity requires the direct sum of the representations with opposite elicity. The photon exists in the two states of elicity ± 1.

We will define the generic state $|\mathbf{p}\rangle$ with $|\mathbf{p}| = |\bar{\mathbf{p}}|$ through a rotation starting from the state $|\bar{\mathbf{p}}\rangle$. The rotation sending $\bar{\mathbf{p}}$ into $\mathbf{p}$ is undetermined on the right for a rotation around the direction of $\bar{\mathbf{p}}$ and on the left for a rotation around the direction of $\mathbf{p}$. We will choose $|\mathbf{p}\rangle$ adopting a standard convention for the Euler angles that define it, i.e.

$$R_{\mathbf{p}} = R_z(\varphi)R_y(\theta)R_z(-\varphi), \tag{5.125}$$

where θ and φ are the polar angles of $\mathbf{p}$. This convention is equivalent to define $R_{\mathbf{p}}$ as a rotation of θ around $\mathbf{n}_3 \wedge \hat{\mathbf{p}}$, with $\mathbf{n}_3$ the versor along the 3 axis and $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$.

With this convention

$$|\mathbf{p}\rangle = U(R_{\mathbf{p}})|\bar{\mathbf{p}}\rangle. \tag{5.126}$$

It is easy to verify that $|\mathbf{p}\rangle$ has the same elicity, a, of $|\bar{\mathbf{p}}\rangle$. In fact

$$\frac{\mathbf{J}\cdot\mathbf{p}}{|\mathbf{p}|}|\mathbf{p}\rangle = U(R_{\mathbf{p}})U^{-1}(R_{\mathbf{p}})\frac{\mathbf{J}\cdot\mathbf{p}}{|\mathbf{p}|}U(R_{\mathbf{p}})|\bar{\mathbf{p}}\rangle = U(R_{\mathbf{p}})\frac{\mathbf{J}\cdot\mathbf{p}}{|\mathbf{p}|}|\bar{\mathbf{p}}\rangle = a|\mathbf{p}\rangle. \tag{5.127}$$

We will define the state $|\bar{\mathbf{p}}'\rangle$ with $\bar{p}'_\mu$ of the form (5.110) and $\bar{p}' \neq \bar{p}$ through the transformation

$$|\bar{\mathbf{p}}'\rangle = U(\Lambda_{\bar{p}'})|\bar{\mathbf{p}}\rangle, \tag{5.128}$$

where $U(\Lambda_{\bar{p}'})$ is a pure velocity transformation along the 3 axis which sends $\bar{\mathbf{p}}$ into $\bar{\mathbf{p}}'$ without rotations around $\bar{\mathbf{p}}$ or $\bar{\mathbf{p}}'$. The rotated states of $|\bar{\mathbf{p}}'\rangle$ will be defined with the convention (5.126).

Once fixed the base in this way let us now construct the representation. If $U(\Lambda)$ is the representative of the generic Lorentz transformation sending $\mathbf{p}$ into $\mathbf{p}'$

$$U(\Lambda)|\mathbf{p}\rangle = U(\Lambda)U(R_{\mathbf{p}})|\bar{\mathbf{p}}\rangle = U(R_{\mathbf{p}'})U^\dagger(R_{\mathbf{p}'})U(\Lambda)U(R_{\mathbf{p}})U^\dagger(\Lambda_{\bar{p}'})U(\Lambda_{\bar{p}'})|\bar{\mathbf{p}}'\rangle. \tag{5.129}$$

It is easy to see that

$$\mathcal{U} = R_{\mathbf{p}'}^{-1}\Lambda R_{\mathbf{p}}\Lambda_{\bar{p}'}^{-1}, \tag{5.130}$$

is a transformation that leaves $\bar{p}'_\mu$ unchanged, i.e. an element of the small group of $\bar{p}'_\mu$. The algebra of such a group is formed by the generators $\epsilon_{\mu\nu\rho\sigma}J^{(\nu\rho)}\bar{p}'^\sigma$, i.e. $\mathbf{J}\cdot\bar{\mathbf{p}}'/|\bar{\mathbf{p}}'|, \Gamma^+, \Gamma^-$. Now Γ^+ and Γ^- are identically zero in the representation under exam, thus $\mathcal{U}$ is a rotation around the 3 axis of a well defined angle $\bar{\varphi}$.

A rotation of an angle $\bar{\varphi}$ around the 3 axis is represented by

$$R_3(\bar{\varphi})|\bar{\mathbf{p}}'\rangle = e^{ia\bar{\varphi}}|\bar{\mathbf{p}}'\rangle, \tag{5.131}$$

where a is the elicity. Then Eq. (5.129) becomes

$$U(\Lambda)|\mathbf{p}\rangle = e^{ia\bar{\varphi}}|\Lambda\mathbf{p}\rangle. \tag{5.132}$$

If Λ is an infinitesimal rotation of parameter $\delta\boldsymbol{\theta}$ we find

$$\begin{aligned} 1 + i\bar{\varphi}J_3 &\approx e^{i\theta\mathbf{J}\cdot(\mathbf{n}_3\wedge\hat{\mathbf{p}})}(1 + i\delta\boldsymbol{\theta}\cdot\mathbf{J})e^{-i\theta\mathbf{J}\cdot(\mathbf{n}_3\wedge\hat{\mathbf{p}}')} \\ &\approx 1 + i\delta\boldsymbol{\theta}\cdot\mathbf{J} - \theta[\mathbf{J}\cdot(\mathbf{n}_3\wedge\hat{\mathbf{p}}), \delta\boldsymbol{\theta}\cdot\mathbf{J}] \\ &\approx 1 + i\delta\boldsymbol{\theta}\cdot\mathbf{J} - i\theta[(\mathbf{J}\cdot\hat{\mathbf{p}})\delta\theta_3 - J_3(\hat{\mathbf{p}}\cdot\delta\boldsymbol{\theta})], \end{aligned} \tag{5.133}$$

where in the first equality we used the fact that $\Lambda_{\bar{p}'} = 1$, in the second the fact that for infinitesimal rotations we may choose $\mathbf{p}' \approx \mathbf{p}$ in the second exponential, and in the third the use of the infinitesimal rotations. We then find

$$\bar{\varphi} = \delta\theta_3(1 - \hat{p}_3\theta) + \delta\boldsymbol{\theta}\cdot\hat{\mathbf{p}}\theta, \tag{5.134}$$

and choosing $\theta = |\mathbf{p}|/(|\mathbf{p}| + p_3)$

$$\bar{\varphi} = \delta\boldsymbol{\theta}\cdot\frac{\mathbf{p} + |\mathbf{p}|\mathbf{n}_3}{|\mathbf{p}| + p_3}. \tag{5.135}$$

Analogously if Λ is an infinitesimal Lorentz transformation of parameter $\delta\boldsymbol{\beta}$ we find

$$\bar{\varphi} = \frac{\delta\beta_1 p_2 - \delta\beta_2 p_1}{|\mathbf{p}| + p_3}. \tag{5.136}$$

The generic state of the particle is written as

$$|\Phi\rangle = \int d\Omega_{\mathbf{p}}\, \Phi(\mathbf{p})|\mathbf{p}\rangle, \tag{5.137}$$

with the scalar product

$$\langle\Phi'|\Phi\rangle = \int d\Omega_{\mathbf{p}}\, \Phi'^*(\mathbf{p})\Phi(\mathbf{p}). \tag{5.138}$$

For a generic Lorentz transformation

$$U(\Lambda)|\Phi\rangle = \int d\Omega_{\mathbf{p}}\, \Phi(\mathbf{p})e^{ia\bar{\varphi}(\Lambda,p)}|\Lambda\mathbf{p}\rangle = \int d\Omega_{\mathbf{p}}\, \Phi(\Lambda^{-1}\mathbf{p})e^{ia\bar{\varphi}(\Lambda,\Lambda^{-1}p)}|\mathbf{p}\rangle. \tag{5.139}$$

The generators on the space of the $\Phi(\mathbf{p})$ functions are

$$\mathbf{J} = -i\mathbf{p}\wedge\frac{\partial}{\partial\mathbf{p}} + \mathbf{s}, \tag{5.140}$$

$$\mathbf{K} = ip^0\frac{\partial}{\partial\mathbf{p}} + \boldsymbol{\chi}, \tag{5.141}$$

with

$$s_1 = a\frac{p_1}{|\mathbf{p}|+p_3} \quad s_2 = a\frac{p_2}{|\mathbf{p}|+p_3} \quad s_3 = a, \tag{5.142}$$

$$\chi_1 = a\frac{p_2}{|\mathbf{p}|+p_3} \quad \chi_2 = -a\frac{p_1}{|\mathbf{p}|+p_3} \quad \chi_3 = 0. \tag{5.143}$$

This generators obey the commutation relations of the algebra (5.28)-(5.30) and are hermitian with the metric (5.138).

This completes the construction of the group representation on the Hilbert space of functions $\Phi(\mathbf{p})$ for a zero mass particle.

4. The Wigner rotation

We here want to calculate explicitly the Wigner rotation for a finite Lorentz transformation, for a massive particle. The velocity transformation is written as

$$U(\Lambda) = e^{-i\mathbf{K}\cdot\mathbf{y}}, \tag{5.144}$$

with

$$\mathbf{K} = ip_0\frac{\partial}{\partial\mathbf{p}} + \frac{\mathbf{p}\wedge\mathbf{s}}{p_0+m}. \tag{5.145}$$

For zero spin

$$e^{\mathbf{y}\cdot p_0\frac{\partial}{\partial\mathbf{p}}}\varphi(\mathbf{p}) = \varphi(\Lambda^{-1}\mathbf{p}). \tag{5.146}$$

For non-zero spin

$$U(\Lambda)\varphi(\mathbf{p}) = e^{\mathbf{y}\cdot p_0\frac{\partial}{\partial\mathbf{p}} - i\mathbf{y}\cdot\frac{\mathbf{p}\wedge\mathbf{s}}{p_0+m}}\varphi(\mathbf{p}), \tag{5.147}$$

where φ has $2s+1$ components. The operator $U(\Lambda)$ is the exponential of two operators which do not commute.

In general given two operators A and B one has

$$e^{A+B} = e^A\sum_{n=0}^{\infty}\int_0^1 dx_1\cdots dx_n\, T(B(x_1)\cdots B(x_n)), \tag{5.148}$$

where $B(x) = e^{-xA}Be^{xA}$ and T is the usual time ordered product

$$T(B(x_1)\cdots B(x_n)) = \frac{1}{n!}\sum_{\substack{\text{permutations}\\ \text{of } \{i_k\}}}\theta(x_{i_1}-x_{i_2})\cdots\theta(x_{i_{n-1}}-x_n)B(x_1)\cdots B(x_n). \tag{5.149}$$

If A and B commute $B(x) = B$ and Eq. (5.148) gives $e^{A+B} = e^Ae^B$. Eq. (5.148) can be proved observing that $U(\lambda) = e^{\lambda(A+B)}$ obeys the equation

$$\frac{d}{d\lambda}U(\lambda) = (A+B)U(\lambda) \quad U(0) = 1. \tag{5.150}$$

Let

$$W(\lambda) = e^{\lambda A}\sum_{n=0}^{\infty}\int_0^{\lambda} dx_1\cdots dx_n\, T(B(x_1)\cdots B(x_n)). \tag{5.151}$$

One easily verifies that

$$\frac{d}{d\lambda}W(\lambda) = (A+B)W(\lambda) \tag{5.152}$$

with $W(0) = 1$. The we must have $U(\lambda) = W(\lambda)$ and for $\lambda = 1$ Eq. (5.148) is recovered.

Let now

$$A = \mathbf{y}\cdot p_0\frac{\partial}{\partial \mathbf{p}} \quad B = -i\mathbf{y}\cdot\frac{\mathbf{p}\wedge\mathbf{s}}{p_0+m}. \tag{5.153}$$

We will have

$$B(x) = e^{-x\mathbf{y}\cdot p_0\frac{\partial}{\partial \mathbf{p}}}B(\mathbf{p})e^{x\mathbf{y}\cdot p_0\frac{\partial}{\partial \mathbf{p}}} = B(\lambda_{-x}^{-1}\mathbf{p}), \tag{5.154}$$

where Λ_x is the Lorentz transformation with parameter $x\mathbf{y}$. In the numerator of B, due to the vector products, only enters the component of $\mathbf{p}$ orthogonal to $\mathbf{y}$ and this is invariant under the transformation. So

$$B(x) = -i\mathbf{y}\cdot\frac{\mathbf{p}\wedge\mathbf{s}}{\Lambda_{-x}^{-1}p_0+m}. \tag{5.155}$$

The $B(x)$ all commute with themselves and

$$\int_0^1 dx_1\cdots dx_n\, T(B(x_1)\cdots B(x_n)) = \frac{1}{n!}\left[\int_0^1 B(x)\,dx\right]^n, \tag{5.156}$$

and

$$U(\Lambda) = e^{\mathbf{y}\cdot p_0\frac{\partial}{\partial \mathbf{p}}}e^{\int_0^1 b(x)\,dx}, \tag{5.157}$$

Moreover

$$U(\Lambda)\varphi(\mathbf{p}) = e^{-i\mathbf{y}\cdot\int_0^1 dx\,\frac{\mathbf{p}\wedge\mathbf{s}}{\Lambda_{1-x}^{-1}p_0+m}}\varphi(\Lambda^{-1}\mathbf{p}), \tag{5.158}$$

Then the Wigner rotation is

$$e^{i\mathbf{s}\cdot(\mathbf{p}\wedge\mathbf{y})\int_0^1\frac{dx}{\Lambda_{1-x}^{-1}p_0+m}}. \tag{5.159}$$

The integral can easily be evaluated if we parametrize p_μ in the form $\mathbf{p}_\perp$, $p_0 = m_t\cosh y_0$, $p_\parallel = m_t\sinh y_0$, and $m_t = \sqrt{m^2+\mathbf{p}_\perp^2}$. Using this parametrization we find

$$\begin{aligned}\int_0^1\frac{dx}{\Lambda_{1-x}^{-1}p_0+m} &= \int_0^1\frac{dx}{m_t\cosh[y_0+y(1-x)]+m}\\ &= \frac{1}{y}\int_0^y\frac{dz}{m_t\cosh(y_0+z)+m}\\ &= \frac{1}{yp_\perp}\varphi,\\ \varphi &= \arcsin\left[\frac{m_t+m\cosh(y_0+z)}{m+m_t\cosh(y_0+z)}\right]_{z=0}^{z=y},\end{aligned} \tag{5.160}$$

is the angle of the Wigner rotation.

5. Discrete transformations

We want now to discuss the discrete transformations, specifically the spatial inversion and the time reversal.

The spatial inversion Π sends

$$\mathbf{p}\to-\mathbf{p} \quad \mathbf{J}\to\mathbf{J} \quad \mathbf{K}\to-\mathbf{K}. \tag{5.161}$$

We immediately find a representation

$$\Pi|\mathbf{p}\rangle = \eta|-\mathbf{p}\rangle, \tag{5.162}$$

and on the wave functions

$$\varphi_a(\mathbf{p}) \to -\eta\varphi_a(-\mathbf{p}), \tag{5.163}$$

where η is a phase factor which must be ± 1 since $\Pi^2 = 1$. It is easy to show that the transformation (5.163) is unitary

$$\begin{aligned}\langle \Pi a'|\Pi a\rangle &= \int d\Omega_{\mathbf{p}} \varphi_a'^{\dagger}(-\mathbf{p})\varphi_a(-\mathbf{p}) \\ &= \int d\Omega_{\mathbf{p}} \varphi_a'^{\dagger}(\mathbf{p})\varphi_a(\mathbf{p}) = \langle a'|a\rangle,\end{aligned} \tag{5.164}$$

Moreover $\langle \Pi a'|\mathbf{p}\Pi a\rangle = -\langle a'|\mathbf{p}a\rangle$ or

$$\Pi^\dagger \mathbf{p}\Pi = -\mathbf{p}, \tag{5.165}$$

and

$$\Pi^\dagger \mathbf{J}\Pi = \mathbf{J}, \tag{5.166}$$

$$\Pi^\dagger \mathbf{K}\Pi = -\mathbf{K}, \tag{5.167}$$

since we assumed η independent of $\mathbf{p}$.

A representation of the time reversal T is in terms of the antiunitary operator

$$\varphi(\mathbf{p}) \to \eta_T C\varphi^*(-\mathbf{p}), \tag{5.168}$$

where the unitary matrix C is defined in Section IV.F and η_T is a phase independent of $\mathbf{p}$. So that

$$\begin{aligned}\langle a'|T^\dagger \mathbf{p}Ta\rangle &= \langle Ta|\mathbf{p}Ta'\rangle = \int d\Omega_{\mathbf{p}} \varphi_a^{\mathrm{Tr}}(-\mathbf{p})\mathbf{p}\varphi_{a'}^*(-\mathbf{p}) \\ &= -\int d\Omega_{\mathbf{p}} \varphi_{a'}^{\dagger}(\mathbf{p})\mathbf{p}\varphi_a(\mathbf{p}) = -\langle a'|\mathbf{p}a\rangle\end{aligned} \tag{5.169}$$

or

$$T^\dagger \mathbf{p}T = -\mathbf{p}, \tag{5.170}$$

Similarly

$$\begin{aligned}\langle a'|T^\dagger \mathbf{J}Ta\rangle &= \langle Ta|\mathbf{J}Ta'\rangle = \int d\Omega_{\mathbf{p}} \varphi_a^{\mathrm{Tr}}(-\mathbf{p})C^*\left(-i\mathbf{p}\wedge\frac{\partial}{\partial\mathbf{p}} + \mathbf{s}\right)C^{\mathrm{Tr}}\varphi_{a'}^*(-\mathbf{p}) \\ &= \int d\Omega_{\mathbf{p}} \varphi_{a'}^{\dagger}(-\mathbf{p})\left(i\mathbf{p}\wedge\frac{\partial}{\partial\mathbf{p}} - \mathbf{s}\right)\varphi_a(-\mathbf{p}),\end{aligned} \tag{5.171}$$

where we integrated by parts and used $C\mathbf{s}^{\mathrm{Tr}}C^\dagger = C\mathbf{s}^*C^\dagger = -\mathbf{s}$ (see Eq. (4.85)). So $\langle Ta|\mathbf{J}Ta'\rangle = -\langle a'|\mathbf{J}a\rangle$ or

$$T^\dagger \mathbf{J}T = -\mathbf{J}. \tag{5.172}$$

Similarly

$$T^\dagger \mathbf{K}T = \mathbf{K}. \tag{5.173}$$

B. Wave functions in coordinate space

In relativistic mechanics the coordinates, $x = (x^0, x^1, x^2, x^3) = (t, \mathbf{x})$, play a privileged role. The constant speed of light principle, together with the relativity principle, implies that a signal cannot propagate at a speed greater than c. This implies, for example, that the regions with $x^2 < 0$ are causally disconnected from the events at $x = (0, \mathbf{0})$. This statement is simple in coordinate space but it does not have an equally explicit expression in other representations.

It is then convenient to associate to a state a wave function $\psi(x)$ which describes the state point by point in space-time. For the description to be effectively linked to the point event it is necessary that $\psi(x)$ transforms *locally*.

For a Lorentz transformations Λ this means

$$\psi(x) \xrightarrow{\Lambda} \psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x), \tag{5.174}$$

or

$$\psi'(\Lambda x) = S(\Lambda)\psi(x), \tag{5.175}$$

where $S(\Lambda)$ does not depend on the point and it is a representation of the Lorentz group.

For a translation, a, we require

$$\psi(x) \xrightarrow{a} \psi'(x) = \psi(x+a). \tag{5.176}$$

Introducing

$$p_\mu \psi(x) = i\frac{\partial}{\partial x^\mu}\psi(x), \tag{5.177}$$

the momentum eigenstate $\psi_{\mathbf{p}}(x)$ can be written as follows

$$\psi_{\mathbf{p}}(x) = e^{-ipx}\psi_{\mathbf{p}}(0), \tag{5.178}$$

where in the exponent we use the simplified notation $px \equiv p_\mu x^\mu$.

If the time evolution is local we will need that $\psi(x)$ obeys to a partial differential equation with derivatives of *finite order*. In what follows we will try to build local wave functions for spin $0, 1/2, 1$ particles. Of course the states of these particles are defined by the unitary irreducible representations of the Poincaré group. Our wave functions will have to be in bijective correspondence with the vectors of such representations, and the scalar product for such vectors will have to be expressible in terms of wave functions. Relative to this metric of the Hilbert space the symmetry transformations on the $\psi(x)$ will have to be unitary. We will verify that the representations of the Lorentz group $S(\Lambda)$ will necessarily be finite dimensional.

We conclude observing that

$$\psi'(0) = S(\Lambda)\psi(0), \tag{5.179}$$

in fact Λ is the small group of point $x = 0$. If we call $U(\Lambda)$ the unitary operator which represents the Lorentz transformation Λ we will have

$$U(\Lambda)\psi(x) = U(\Lambda)e^{-ipx}\psi(0) = U(\Lambda)e^{-ipx}U^{-1}(\Lambda)U(\Lambda)\psi(0), \tag{5.180}$$

but

$$U(\Lambda)e^{-ipx}U^{-1}(\Lambda) = e^{-i(\Lambda p)x} = e^{-ip(\Lambda^{-1}x)}, \tag{5.181}$$

and, since $U(\Lambda)\psi(0) = S(\Lambda)\psi(0)$,

$$U(\Lambda)\psi(x) = e^{-ip(\Lambda^{-1}x)}S(\Lambda)\psi(0) = S(\Lambda)\psi(\Lambda^{-1}x). \tag{5.182}$$

VI. THE RELATIVISTIC WAVE EQUATIONS

In Section V we introduced the Poincaré group and showed that a structureless particle is described by a unitary irreducible representation of this group identified by the mass and by the spin. We will now find the relativistic wave equations of free motion for these particles.

A. Particles of spin 0

For a spin 0 particle any given state $|s\rangle$ can be represented as

$$|s\rangle = \int d\Omega_{\mathbf{p}}\varphi_s(\mathbf{p})|\mathbf{p}\rangle, \tag{6.1}$$

$\varphi_s(\mathbf{p}) = \langle \mathbf{p}|s\rangle$ is the wave function in the representation were the momenta are diagonal. The wave function associated to the state $|\mathbf{p}\rangle$, $\psi_{\mathbf{p}}(x)$, must have the form (5.178). By the superposition principle

$$\langle x|s\rangle = \psi_s(x) = \int d\Omega_{\mathbf{p}} \varphi_s(\mathbf{p}) e^{-ipx} \psi_{\mathbf{p}}(0). \tag{6.2}$$

To determine $\psi_{\mathbf{p}}(0)$ let us consider the effect of a Lorentz transformation

$$|s\rangle \xrightarrow{\Lambda} \int d\Omega_{\mathbf{p}}\, \varphi_s(\mathbf{p}) |\Lambda\mathbf{p}\rangle = \int d\Omega_{\mathbf{p}}\, \varphi_s(\Lambda^{-1}\mathbf{p})|\mathbf{p}\rangle, \tag{6.3}$$

and on the wave function

$$\psi_s(x) \xrightarrow{\Lambda} \int d\Omega_{\mathbf{p}}\, \varphi_s(\Lambda^{-1}\mathbf{p}) e^{-ipx} \psi_{\mathbf{p}}(0) = \int d\Omega_{\mathbf{p}}\, \varphi_s(\mathbf{p}) e^{-ip(\Lambda^{-1}x)} \psi_{\Lambda\mathbf{p}}(0). \tag{6.4}$$

This transformation is certainly local if $\psi_{\Lambda\mathbf{p}}(0) = \psi_{\mathbf{p}}(0)$. This means that $\psi_{\mathbf{p}}(0)$ must be an invariant constructed with p_μ. Since $p^2 = m^2$, such an invariant must be a constant, that can be chosen equal to 1.

So

$$\psi_s(x) = \int d\Omega_{\mathbf{p}}\, \varphi_s(\mathbf{p}) e^{-ipx}. \tag{6.5}$$

Under translation

$$\psi_s(x) \xrightarrow{a} \psi'(x) = \psi_s(x+a). \tag{6.6}$$

Under Lorentz transformation

$$\psi_s(x) \xrightarrow{\Lambda} \psi'(x) = \psi_s(\Lambda^{-1}x). \tag{6.7}$$

The function $\psi_s(x)$ in Eq. (6.5) transforms locally and the requirement $p^2 = m^2$ implies that it obeys the Klein-Gordon equation

$$(\Box + m^2)\psi_s(x) = 0, \tag{6.8}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial x^{i\,2}}, \tag{6.9}$$

is the d' Alambert operator. Eq. (6.8) is invariant under transformations of the Poincaré group.

Not all solutions of Eq. (6.8) are of kind (6.5). Eq. (6.8) admits also solutions with negative energy. As a matter of fact the wave function of Eq. (6.5) obeys to the following equation

$$\left(i\frac{\partial}{\partial x^0} - \sqrt{m^2 - \boldsymbol{\nabla}^2} \right) \psi_s(x) = 0, \tag{6.10}$$

which is non-local. The requirement for a local equation imposes to have negative energy solutions as well.

The general solution of Eq. (6.8) can be easily obtained working in Fourier space

$$\psi_s(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx} \tilde{\psi}_s(p). \tag{6.11}$$

Then Eq. (6.8) becomes

$$(p^2 - m^2)\tilde{\psi}_s(p) = 0, \tag{6.12}$$

or

$$\tilde{\psi}_s(p) = \varphi_s(p) 2\pi \delta(p^2 - m^2). \tag{6.13}$$

Integrating over p_0 in Eq. (6.11)

$$\psi_s(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p_0} \left[\varphi_s(\mathbf{p}, p_0) e^{-ipx} + \varphi_s(\mathbf{p}, -p_0) e^{ip_0 x^0 + i\mathbf{p}\cdot\mathbf{r}}\right]. \tag{6.14}$$

We then define

$$\varphi_s(\mathbf{p}, p_0) = \varphi_s^+(\mathbf{p}), \tag{6.15}$$

$$\varphi_s(\mathbf{p}, -p_0) = \varphi_s^-(-\mathbf{p}), \tag{6.16}$$

so that

$$\psi_s(x) = \int d\Omega_{\mathbf{p}} \left[\varphi_s^+(\mathbf{p}) e^{-ipx} + \varphi_s^-(\mathbf{p}) e^{ipx}\right]. \tag{6.17}$$

A natural scalar product can be introduced as follows. Given two solutions of the Klein-Gordon equation (6.8), $\psi_a(x)$ and $\psi_b(x)$, the quantity

$$J_\mu^{(a,b)}(x) = i\psi_a^* \overleftrightarrow{\partial}_\mu \psi_b = i\left[\psi_a^* \partial_\mu \psi_b - (\partial_\mu \psi_a^*)\psi_b\right], \tag{6.18}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$, is conserved, i.e.

$$\partial^\mu J_\mu^{(a,b)}(x) = 0. \tag{6.19}$$

Then, due to Gauss theorem, if the ψ go to zero sufficiently rapidly at spatial infinity, the integral extended to an hypersurface of spatial kind extended to infinity,

$$\int d\sigma^\mu J_\mu^{(a,b)}(x), \tag{6.20}$$

is independent from the surface ($d\sigma^\mu$ is the oriented normal). It can be calculated on a surface $x^0 =$ constant

$$\int d\sigma^\mu J_\mu^{(a,b)}(x) = \int d\mathbf{x} J_0^{(a,b)}(t, \mathbf{x}). \tag{6.21}$$

We will define the scalar product $\langle a|b\rangle$ through

$$\begin{aligned} \langle a|b\rangle &= \int d\sigma^\mu J_\mu^{(a,b)}(x) \\ &= \int d\Omega_{\mathbf{p}} \left[\varphi_a^{+*}(\mathbf{p})\varphi_b^+(\mathbf{p}) - \varphi_a^{-*}(\mathbf{p})\varphi_b^-(\mathbf{p})\right]. \end{aligned} \tag{6.22}$$

The generators of the group in this representation are

$$\begin{aligned} p_\mu &= i\frac{\partial}{\partial x^\mu}, \\ J^{(\mu\nu)} &= -i\left(x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu}\right), \end{aligned} \tag{6.23}$$

which are hermitians under the metric of Eq. (6.22).

The Eq. (6.10) satisfied by these wave functions is non-local. In order to have a local equation, like (6.8), it is necessary to put together positive and negative energy solutions. Actually, the Klein-Gordon Eq. (6.8) is second order in the temporal derivative, while, once the Hamiltonian is known, the evolution equation should be of the first order.

B. Particles of spin 1/2

The irreducible representations of the Poincaré group corresponding to particles of mass m and spin 1/2 are in correspondence with vectors $|r, \mathbf{p}\rangle$, where r is the eigenvalue of one component of the spin in the rest frame.

Any state $|a\rangle$ of the Hilbert space generated like so is of the following form

$$|a\rangle = \int d\Omega_{\mathbf{p}} \sum_{r=1}^{2} \varphi_a(r,\mathbf{p})|r,\mathbf{p}\rangle. \tag{6.24}$$

The infinitesimal transformations of the Lorentz group are

$$\varphi_a(r,\mathbf{p}) \to \varphi_{\Lambda a}(r,\mathbf{p}) = \left[1 + i\boldsymbol{\theta}\cdot\left(\frac{1}{i}\mathbf{p}\wedge\frac{\partial}{\partial\mathbf{p}} + \mathbf{s}\right) - i\boldsymbol{\alpha}\cdot\left(ip_0\frac{\partial}{\partial\mathbf{p}} + \frac{\mathbf{p}\wedge\mathbf{s}}{p_0+m}\right)\right]_{rr'}\varphi_a(r',\mathbf{p}). \tag{6.25}$$

We will now construct local wave functions for these states. The locality under translations fixes the form of the wave functions corresponding to eigenstates of momentum

$$\psi_{r,\mathbf{p}}(x) = \psi_{r,\mathbf{p}}(0)e^{-ipx}. \tag{6.26}$$

We will call $\psi_{r,\mathbf{p}}(0) \equiv u(r,\mathbf{p})$. Due to the superposition principle we will have

$$\psi_a(x) = \int d\Omega_{\mathbf{p}} \sum_{r=1}^{2} \varphi_a(r,\mathbf{p})u(r,\mathbf{p})e^{-ipx}. \tag{6.27}$$

To find an explicit form for the local wave functions we will adopt the following strategy. We will assume specific properties of local transformations for $\psi_a(x)$. We will write an equation explicitly covariant under the Poincaré group transformations and will later prove that the solutions of this equation give a unitary representation of the Poincaré group. And will express the scalar product between states in terms of these wave functions.

Locality under group transformations requires

$$\psi_a(x) \xrightarrow{\Lambda} \psi_{\Lambda a}(x) = S(\Lambda)\psi_a(\Lambda^{-1}x), \tag{6.28}$$

where $S(\Lambda)$ is a finite dimensional representation of the Lorentz group. The Lorentz group is locally isomorphic to $SU(2)\otimes SU(2)$. Hence the finite dimensional representations are fixed by two numbers (n_+, n_-) which determine the representations of the two groups $SU(2)$ with generators

$$\mathbf{J}_+ = \frac{\mathbf{J}+i\mathbf{K}}{2}, \tag{6.29}$$

$$\mathbf{J}_- = \frac{\mathbf{J}-i\mathbf{K}}{2}. \tag{6.30}$$

We will heuristically construct the ψ with representations of dimension 2.

There exist two inequivalent representations of dimension 2. The $(\frac{1}{2},0)$ and the $(0,\frac{1}{2})$. In the two representations the group generators, defined by the infinitesimal transformations

$$\boldsymbol{\Lambda} \approx 1 + i\boldsymbol{\theta}\cdot\mathbf{J} - i\boldsymbol{\alpha}\cdot\mathbf{K}, \tag{6.31}$$

are given by

$$(\frac{1}{2},0) \;:\; \begin{cases}\mathbf{J} = \frac{\boldsymbol{\sigma}}{2}\\ \mathbf{K} = -i\frac{\boldsymbol{\sigma}}{2}\end{cases}, \tag{6.32}$$

$$(0,\frac{1}{2}) \;:\; \begin{cases}\mathbf{J} = \frac{\boldsymbol{\sigma}}{2}\\ \mathbf{K} = i\frac{\boldsymbol{\sigma}}{2}\end{cases}. \tag{6.33}$$

The corresponding finite transformations are

$$S_{(\frac{1}{2},0)}(\Lambda) = e^{i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\alpha}\cdot\frac{\boldsymbol{\sigma}}{2}}, \tag{6.34}$$

$$S_{(0,\frac{1}{2})}(\Lambda) = e^{i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\alpha}\cdot\frac{\boldsymbol{\sigma}}{2}}, \tag{6.35}$$

with

$$S_{(0,\frac{1}{2})}(\Lambda) = {S^{\dagger}_{(\frac{1}{2},0)}}^{-1}(\Lambda). \tag{6.36}$$

We will call ξ the spinors which transform according to $(\frac{1}{2},0)$ and η the ones transforming according to $(0,\frac{1}{2})$.
Since $\mathbf{J}$ is an axial vector, whereas $\mathbf{K}$ is a polar vector, we have under parity

$$(\frac{1}{2},0) \overset{P}{\longleftrightarrow} (0,\frac{1}{2}). \tag{6.37}$$

Then in order to construct a representation invariant under parity we need to consider a reducible representation of the Lorentz group for $S(\Lambda)$, namely

$$(\frac{1}{2},0) \oplus (0,\frac{1}{2}). \tag{6.38}$$

The vectorial space for this representation is composed by spinors of 4 components of the form

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{6.39}$$

In such representation, the generators of the Lorentz group are

$$\mathbf{J} \;=\; \begin{pmatrix} \frac{\boldsymbol{\sigma}}{2} & 0 \\ 0 & \frac{\boldsymbol{\sigma}}{2} \end{pmatrix}, \tag{6.40}$$

$$\mathbf{K} \;=\; \begin{pmatrix} -i\frac{\boldsymbol{\sigma}}{2} & 0 \\ 0 & i\frac{\boldsymbol{\sigma}}{2} \end{pmatrix}. \tag{6.41}$$

Using the following identities for the Pauli matrices

$$[\boldsymbol{\theta}\cdot\boldsymbol{\sigma},\boldsymbol{\sigma}] \;=\; -2i\boldsymbol{\theta}\wedge\boldsymbol{\sigma}, \tag{6.42}$$

$$\{\boldsymbol{\alpha}\cdot\boldsymbol{\sigma},\boldsymbol{\sigma}\} \;=\; 2\boldsymbol{\alpha}, \tag{6.43}$$

where $[\ldots]$ stands for the commutator and $\{\ldots\}$ for the anticommutator, we easily find that

$$\begin{aligned} S_{(\frac{1}{2},0)}(\Lambda)(p_0+\mathbf{p}\cdot\boldsymbol{\sigma})S^\dagger_{(\frac{1}{2},0)}(\Lambda) &= p_0+\mathbf{p}\cdot\boldsymbol{\sigma}+i[\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2},p_0+\mathbf{p}\cdot\boldsymbol{\sigma}]-\{\boldsymbol{\alpha}\cdot\frac{\boldsymbol{\sigma}}{2},p_0+\mathbf{p}\cdot\boldsymbol{\sigma}\}+\ldots \\ &= p_0+\mathbf{p}\cdot\boldsymbol{\sigma}+\frac{i}{2}[\boldsymbol{\theta}\cdot\boldsymbol{\sigma},\mathbf{p}\cdot\boldsymbol{\sigma}]-\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}p_0-\frac{1}{2}\{\boldsymbol{\alpha}\cdot\boldsymbol{\sigma},\mathbf{p}\cdot\boldsymbol{\sigma}\}+\ldots \\ &= (p_0-\boldsymbol{\alpha}\cdot\mathbf{p}+\ldots)+\boldsymbol{\sigma}\cdot(\mathbf{p}-\boldsymbol{\theta}\wedge\mathbf{p}-\boldsymbol{\alpha}p_0+\ldots) \\ &= p_0'+\boldsymbol{\sigma}\cdot\mathbf{p}', \end{aligned} \tag{6.44}$$

where in the vector representation we used

$$(iJ_i)_{jk} \;=\; \epsilon_{ijk}, \tag{6.45}$$

$$(iK_i)_{j0} \;=\; (iK_i)_{0j}=\delta_{ij}, \tag{6.46}$$

and p' is the Lorentz transformed of p

$$p'_\mu=\Lambda^\nu{}_\mu p_\nu. \tag{6.47}$$

Then an equation of the form

$$(p^0+\mathbf{p}\cdot\sigma)\eta=c\xi, \tag{6.48}$$

where c is a scalar, is covariant under Lorentz transformations. In fact, calling $S=S_{(\frac{1}{2},0)}(\Lambda)$, we have

$$S(p^0+\mathbf{p}\cdot\sigma)\eta=cS\xi, \tag{6.49}$$

or

$$S(p^0+\mathbf{p}\cdot\sigma)S^\dagger {S^\dagger}^{-1}\eta=cS\xi'. \tag{6.50}$$

Due to Eq. (6.36) $S^{\dagger^{-1}}\eta = \eta'$ and using Eq. (6.44)

$$(p^{0'} + \mathbf{p}' \cdot \boldsymbol{\sigma})\eta' = \xi', \tag{6.51}$$

so the equation has the same form in all reference frames. Analogously we show that $(p^0 - \mathbf{p}\cdot\boldsymbol{\sigma})\xi$ transforms as $(0, \frac{1}{2})$. The most general system of first order covariant equations has then the following form

$$(p^0 + \mathbf{p} \cdot \sigma)\eta \;=\; c\xi, \tag{6.52}$$
$$(p^0 - \mathbf{p} \cdot \sigma)\xi \;=\; c'\eta, \tag{6.53}$$

and invariance under parity imposes $c = c'$. Multiplying the first equation by $(p^0 - \mathbf{p}\cdot\boldsymbol{\sigma})$ and using the second equation we find

$${p^0}^2 - \mathbf{p}^2 = c^2. \tag{6.54}$$

Then if we want to describe a particle we must identify c with the mass m. In terms of bispinors we have

$$\begin{pmatrix} 0 & p^0 + \mathbf{p}\cdot\boldsymbol{\sigma} \\ p^0 - \mathbf{p}\cdot\boldsymbol{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = m \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{6.55}$$

We give a more symmetric form to this equation by introducing the 4×4 matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{6.56}$$

and the bispinor $\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$. We will also introduce

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\sigma\tau}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\tau. \tag{6.57}$$

We then find

$$(\gamma^0 p^0 - \boldsymbol{\gamma}\cdot\mathbf{p})\psi = m\psi, \tag{6.58}$$

or

$$\gamma^\mu p_\mu \psi = m\psi. \tag{6.59}$$

Introducing the notation $\not{p} = \gamma^\mu p_\mu$ we have

$$(\not{p} - m)\psi = 0. \tag{6.60}$$

This equation is known as the Dirac equation.

Applying the Lorentz transformation $S(\Lambda)$ in the representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ to the Dirac equation

$$S(\Lambda)\gamma^\mu p_\mu S^{-1}(\Lambda)S(\Lambda)\psi = mS(\Lambda)\psi. \tag{6.61}$$

Since the bispinor transforms under $S(\Lambda)$ the covariance imposes

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu, \tag{6.62}$$

which means that γ^μ transform as a four-vector.

In coordinate representation

$$(i\not{\partial} - m)\psi = 0, \tag{6.63}$$

and by construction the solutions of this equation transform locally under Lorentz transformations. Of course in order to know whether they represent the states of a spin 1/2 particle of mass m we must verify that they are in bijective correspondence with the states defined in terms of the representations of the Poincaré group, and that a transformation on the states corresponds to a transformation on the wave functions.

We have

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{6.64}$$

we can define the covariant component of the gamma matrices

$$\gamma_\mu = g_{\mu\nu}\gamma^\nu, \tag{6.65}$$

and we find

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{6.66}$$

Also

$$\{\gamma^\mu, \gamma^5\} = 0, \tag{6.67}$$

and

$$\gamma^{0\dagger} = \gamma^0 \quad \gamma^{i\dagger} = -\gamma^i, \tag{6.68}$$

or

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{6.69}$$

Using the matrices γ^μ it is possible to write in a compact form the Lorentz transformations in the representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. Consider the matrices

$$\sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu, \gamma_\nu]. \tag{6.70}$$

Under the transformation $S(\Lambda)\sigma_{\mu\nu}S^{-1}(\Lambda)$ they transform as an antisymmetric tensor of rank 2. One can verify that

$$K^i = \frac{1}{2}\sigma^{oi} \quad J^i = \frac{1}{2}\epsilon^{oijk}\sigma_{jk} \quad i, j, k = 1, 2, 3. \tag{6.71}$$

The tensor $\sigma_{\mu\nu}$ represents the generators of the Lorentz group and we can write

$$S(\Lambda) = e^{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}. \tag{6.72}$$

Moreover $\sigma_{\mu\nu}/2$ satisfies the algebra (5.41).

The matrix γ^0 has the role of exchanging the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, so it coincides with the parity operator up to a phase,

$$\psi_a(x) \xrightarrow{P} \psi_{Pa}(x) = \eta_P\gamma^0\psi_a(x^0, -\mathbf{x}). \tag{6.73}$$

From the anticommutation rules (6.64) follows

$$\gamma^0\gamma^i\gamma^0 = -\gamma^i \quad i = 1, 2, 3 \quad \gamma^0\gamma^0\gamma^0 = \gamma^0. \tag{6.74}$$

It is interesting to consider the set of the 16 matrices

$$1, \gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \sigma^{\mu\nu}. \tag{6.75}$$

From the definition follow that the properties of Lorentz transformation of the matrices (6.75) are

$$\begin{array}{ll} 1 & \text{scalar} \\ \gamma^5 & \text{pseudoscalar} \\ \gamma^\mu & \text{vector} \\ \gamma^5\gamma^\mu & \text{pseudovector} \\ \sigma^{\mu\nu} & \text{antisymmetric tensor} \end{array} \tag{6.76}$$

These 16 matrices are linearly independent (in fact they transform differently under Lorentz transformations) so they constitute a complete basis for the 4×4 matrices, i.e. any 4×4 matrix can be written in the form

$$\sum_{a=1}^{16} c_a \Gamma^a, \tag{6.77}$$

where $\{\Gamma^a\}$ is the set of 16 matrices (6.75).

Note that if ψ and ψ' are two bispinors, $\psi'^\dagger \psi$ is not a scalar density. In fact

$$\psi'^\dagger(x)\psi(x) \xrightarrow{(a,\Lambda)} \psi'^\dagger(\Lambda^{-1}x + a) S^\dagger(\Lambda) S(\Lambda) \psi(\Lambda^{-1}x + a), \tag{6.78}$$

and $S^\dagger S \neq 1$. The representation $S(\Lambda)$ is not unitary as follows from its definition (6.32)-(6.33) and as should be expected since the Lorentz group is not compact. But we have

$$S^\dagger(\Lambda)\gamma^0 = \gamma^0 S^{-1}(\Lambda). \tag{6.79}$$

Then, upon defining $\bar{\psi}' = \psi^\dagger \gamma^0$, $\bar{\psi}'\psi$ is a scalar density

$$\begin{aligned} \bar{\psi}'(x)\psi(x) &\xrightarrow{(a,\Lambda)} \psi'^\dagger(\Lambda^{-1}x + a) S^\dagger(\Lambda)\gamma^0 S(\Lambda)\psi(\Lambda^{-1}x + a) \\ &= \bar{\psi}'(\Lambda^{-1}x + a)\psi(\Lambda^{-1}x + a). \end{aligned} \tag{6.80}$$

Let us finally mention the following formulas,

$$\mathrm{Tr}\{\gamma^{\mu_1}\gamma^{\mu_2} \cdot \gamma^{\mu_{2n+1}}\} = 0, \tag{6.81}$$

$$\mathrm{Tr}\{\gamma^\mu \gamma^\nu\} = 4 g^{\mu\nu}, \tag{6.82}$$

$$\mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4\{g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\}, \tag{6.83}$$

$$\mathrm{Tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = -4i\epsilon^{\mu\nu\rho\sigma}, \tag{6.84}$$

$$\gamma_\mu \not{A} \gamma^\mu = -2\not{A}, \tag{6.85}$$

$$\gamma_\mu \not{A}\not{B}\gamma^\mu = 4AB, \tag{6.86}$$

$$\gamma_\mu \not{A}\not{B}\not{C}\gamma^\mu = -2\not{C}\not{B}\not{A}. \tag{6.87}$$

1. Dirac equation solutions: momentum eigenstates

Multiplying Eq. (6.60) by γ^0 we find

$$p^0\psi = (\boldsymbol{\alpha} \cdot \mathbf{p} + \gamma^0 m)\psi, \tag{6.88}$$

where $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$. Now we do a change of representation where we diagonalize γ^0

$$\psi \to U\psi \quad \gamma^\mu \to U\gamma^\mu U^{-1} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = U^{-1}, \tag{6.89}$$

explicitly

$$U \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \dfrac{\xi + \eta}{\sqrt{2}} \\ \dfrac{\xi - \eta}{\sqrt{2}} \end{pmatrix} \equiv \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \tag{6.90}$$

After this transformation the algebra of the γ matrices remains the same. The γ matrices are rewritten as follows

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.91}$$

Since γ^0 is diagonal in the non-relativistic limit the states in this representation have definite parity. This is known as *Pauli representation*. The one of Eq. (6.56) as *spinorial or Kramers representation*.

Let us now find the solution with definite momentum and positive energy in the form

$$\psi_{\mathbf{p}}(x) = e^{-ipx} u(r, \mathbf{p}), \tag{6.92}$$

suggested by translational invariance.

In the Pauli representation we find then

$$\begin{aligned} p^0 u_1 - \boldsymbol{\sigma} \cdot \mathbf{p} u_2 &= m u_1, & (6.93) \\ -p^0 u_2 + \boldsymbol{\sigma} \cdot \mathbf{p} u_1 &= m u_2, & (6.94) \end{aligned}$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$ and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

These equations admit two independent solutions labeled by two Pauli spinors (bidimensional) w_1 and w_2 orthonormal

$$u(r, \mathbf{p}) = c \begin{pmatrix} w_r \\ \dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p^0 + m} w_r \end{pmatrix} \qquad w_r^\dagger w_s = \delta_{rs}. \tag{6.95}$$

Since we know that $\bar{u} u$ must be invariant, we find

$$\begin{aligned} \bar{u} u = u^\dagger \gamma^0 u &= w^\dagger w_r c^2 \left(1 - \frac{\boldsymbol{\sigma}^\dagger \cdot \mathbf{p} \boldsymbol{\sigma} \cdot \mathbf{p}}{(p^0 + m)^2} \right) \\ &= c^2 \left(1 - \frac{\mathbf{p}^2}{(p^0 + m)^2} \right) \\ &= c^2 \frac{2m}{p^0 + m} = \text{invariant}. \end{aligned} \tag{6.96}$$

We then choose conveniently $c = \sqrt{p^0 + m}$ so that

$$\begin{aligned} u(r, \mathbf{p}) &= \begin{pmatrix} \sqrt{p^0 + m} w_r \\ \dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{p^0 + m}} w_r \end{pmatrix}, & (6.97) \\ \bar{u}(r, \mathbf{p}) u(s, \mathbf{p}) &= 2m \delta_{rs}. & (6.98) \end{aligned}$$

As a standard base for the spinors w_r we can take the eigenstates of σ_z

$$w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6.99}$$

As in the scalar case the Dirac equation admits also negative energy solutions. These will be of the following kind

$$\tilde{\psi}(x) = e^{ip^0 t + i\mathbf{p} \cdot \mathbf{x}} \tilde{u}(r, \mathbf{p}), \tag{6.100}$$

Proceeding as in the previous case we find

$$\tilde{u}(r, \mathbf{p}) = \begin{pmatrix} -\dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{p^0 + m}} \tilde{w}_r \\ \sqrt{p^0 + m} \tilde{w}_r \end{pmatrix}. \tag{6.101}$$

Calling $v(r, \mathbf{p}) = \tilde{u}(r, -\mathbf{p})$ we find

$$\begin{aligned} v(r, \mathbf{p}) &= \begin{pmatrix} \dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{p^0 + m}} \tilde{w}_r \\ \sqrt{p^0 + m} \tilde{w}_r \end{pmatrix}, & (6.102) \\ \bar{v}(r, \mathbf{p}) v(s, \mathbf{p}) &= -2m \delta_{rs}. & (6.103) \end{aligned}$$

The spinors u and v satisfy the following algebraic equations

$$\begin{aligned} (\not{p} - m) u(r, \mathbf{p}) &= 0, & (6.104) \\ (\not{p} + m) v(r, \mathbf{p}) &= 0, & (6.105) \end{aligned}$$

and constitute a complete set of spinors for the description of the momentum eigenstates. The four solutions found form a set of independent vectors, orthogonal respect to the γ^0 metric

$$\begin{aligned}\bar{u}(r,\mathbf{p})u(s,\mathbf{p}) &= 2m\delta_{rs}, &(6.106)\\ \bar{v}(r,\mathbf{p})v(s,\mathbf{p}) &= -2m\delta_{rs}, &(6.107)\\ \bar{u}(r,\mathbf{p})v(s,\mathbf{p}) &= \bar{v}(r,\mathbf{p})u(s,\mathbf{p}) = 0. &(6.108)\end{aligned}$$

Due to the completeness of the set we also have

$$\begin{aligned}\sum_{r=1}^{2} u(r,\mathbf{p})\bar{u}(r,\mathbf{p}) &= \not{p} + m, &(6.109)\\ \sum_{r=1}^{2} v(r,\mathbf{p})\bar{v}(r,\mathbf{p}) &= \not{p} - m. &(6.110)\end{aligned}$$

2. Transformation properties and connection with the Poincaré group representations

We will now explicitly study the effect of the Lorentz transformation $S(\Lambda)$ on the solutions we just found. We will find that they realize a representation of the Poincaré group for a spin 1/2 particle.

A Lorentz transformation sends solutions with momentum p to solutions with momentum $p' = \Lambda p$. In fact, using the covariance property of the γ matrices we find

$$S(\Lambda)(\not{p} - m)u(r,\mathbf{p}) = (\not{p}' - m)S(\Lambda)u(r,\mathbf{p}) = 0. \tag{6.111}$$

In the Pauli representation we find for a rotation $R(\boldsymbol{\theta})$

$$\mathbf{J} = \begin{pmatrix} \frac{\boldsymbol{\sigma}}{2} & 0 \\ 0 & \frac{\boldsymbol{\sigma}}{2} \end{pmatrix} \qquad S(R(\boldsymbol{\theta})) = \begin{pmatrix} e^{i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}} & 0 \\ 0 & e^{i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}} \end{pmatrix}, \tag{6.112}$$

so

$$S(R(\boldsymbol{\theta}))u(r,\mathbf{p}) = \begin{pmatrix} \sqrt{p^0+m}\, e^{i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}} w_r \\ \dfrac{(R^{-1}(\boldsymbol{\theta})\boldsymbol{\sigma})\cdot\mathbf{p}}{\sqrt{p^0+m}} e^{i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}} w_r \end{pmatrix} = \begin{pmatrix} \sqrt{p^0+m}\, e^{i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}} w_r \\ \dfrac{\boldsymbol{\sigma}\cdot(R(\boldsymbol{\theta})\mathbf{p})}{\sqrt{p^0+m}} e^{i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}} w_r \end{pmatrix}, \tag{6.113}$$

and

$$e^{i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}} w_r = \sum_{r'} \mathcal{R}(\boldsymbol{\theta})_{r'r} w_{r'} \quad \mathcal{R}(\boldsymbol{\theta})_{r'r} = \left(e^{i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}}\right)_{r'r} \quad S(R(\boldsymbol{\theta}))u(r,\mathbf{p}) = \sum_{r'} \mathcal{R}(\boldsymbol{\theta})_{r'r} u(r', R\mathbf{p}). \tag{6.114}$$

A transformation of rapidity $\boldsymbol{\alpha}$ is given by

$$S(\Lambda_{\boldsymbol{\alpha}}) = \begin{pmatrix} e^{-\boldsymbol{\alpha}\cdot\frac{\boldsymbol{\sigma}}{2}} & 0 \\ 0 & e^{\boldsymbol{\alpha}\cdot\frac{\boldsymbol{\sigma}}{2}} \end{pmatrix} = \begin{pmatrix} \cosh\frac{\alpha}{2} - \hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}\sinh\frac{\alpha}{2} & 0 \\ 0 & \cosh\frac{\alpha}{2} + \hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}\sinh\frac{\alpha}{2} \end{pmatrix}, \tag{6.115}$$

and in the Pauli representation

$$US(\Lambda_{\boldsymbol{\alpha}})U^{-1} = \begin{pmatrix} \cosh\frac{\alpha}{2} & -\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}\sinh\frac{\alpha}{2} \\ -\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}\sinh\frac{\alpha}{2} & \cosh\frac{\alpha}{2} \end{pmatrix}. \tag{6.116}$$

We then find explicitly

$$\begin{pmatrix} \cosh\frac{\alpha}{2} & -\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}\sinh\frac{\alpha}{2} \\ -\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}\sinh\frac{\alpha}{2} & \cosh\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \sqrt{p^0+m}\,w_r \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\sqrt{p^0+m}} w_r \end{pmatrix} = \begin{pmatrix} \sqrt{p'^0+m}\, e^{-i\varphi\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\alpha}}\wedge\hat{\mathbf{p}}} w_r \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{\sqrt{p'^0+m}} e^{-i\varphi\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\alpha}}\wedge\hat{\mathbf{p}}} w_r \end{pmatrix}, \tag{6.117}$$

where

$$\tan\varphi = \frac{|\mathbf{p}|\sinh\frac{\alpha}{2}}{(p^0+m)\cosh\frac{\alpha}{2} - p_{\parallel}\sinh\frac{\alpha}{2}}. \tag{6.118}$$

Here we used Eqs. (D23) and (D24) and $p_{\parallel} = \hat{\boldsymbol{\alpha}}\cdot\mathbf{p}$. The matrix $\mathcal{R} = e^{-i\varphi\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\alpha}}\wedge\hat{\mathbf{p}}}$ is a rotation of an angle $-2\varphi\hat{\boldsymbol{\alpha}}\wedge\hat{\mathbf{p}}$ which acts on the components of the spinor w. Explicitly

$$S(\Lambda_{\boldsymbol{\alpha}})u(r,\mathbf{p}) = \sum_{r'} \mathcal{R}(\Lambda_{\boldsymbol{\alpha}},\mathbf{p})_{r'r}u(r',\Lambda_{\boldsymbol{\alpha}}\mathbf{p}). \tag{6.119}$$

For an infinitesimal transformation ($\alpha \ll 1$)

$$\varphi \approx \frac{\alpha}{2}\frac{|\mathbf{p}|}{p^0+m}, \tag{6.120}$$

$$\mathcal{R} \approx 1 - i\frac{\boldsymbol{\sigma}}{2}\cdot\frac{\boldsymbol{\alpha}\wedge\mathbf{p}}{p^0+m} = 1 + i\mathbf{s}\cdot\frac{\mathbf{p}\wedge\boldsymbol{\alpha}}{p^0+m}. \tag{6.121}$$

So in general we find

$$S(\Lambda)u(r,\mathbf{p}) = \sum_{r'} \mathcal{R}(\Lambda,\mathbf{p})_{r'r}u(r',\Lambda\mathbf{p}), \tag{6.122}$$

where $\mathcal{R}$ is the Wigner rotation associated to the transformation Λ. And an identical formula holds for $v(r,\mathbf{p})$.

Let us now consider any solution of the Dirac equation

$$\psi(x) = \sum_{r=1}^{2}\int d\Omega_{\mathbf{p}}\left[\varphi_r^+(\mathbf{p})u(r,\mathbf{p})e^{-ipx} + \varphi_r^-(\mathbf{p})v(r,\mathbf{p})e^{ipx}\right]. \tag{6.123}$$

By construction the Poincaré group is realized in a local way on the space of these solutions

$$T_a \;:\quad \psi(x) \xrightarrow{a} \psi'(x) = \psi(x+a), \tag{6.124}$$

$$\Lambda \;:\quad \psi(x) \xrightarrow{\Lambda} \psi'(x) = \psi(\Lambda^{-1}x). \tag{6.125}$$

For infinitesimal transformations, recalling that $\left(\Lambda^{-1}x\right)^{\mu} \approx x^{\mu} - \omega^{\mu}{}_{\nu}x^{\nu}$, we have

$$\psi(x) \xrightarrow{a} (1 + a^{\mu}\partial_{\mu})\psi(x), \tag{6.126}$$

$$\psi(x) \xrightarrow{\Lambda} (1 + \frac{i}{2}\omega^{\mu\nu}\sigma_{\mu\nu} - \omega^{\mu\nu}x_{\nu}\partial_{\mu})\psi(x). \tag{6.127}$$

And the generators are

$$p_{\mu} = i\partial_{\mu}, \tag{6.128}$$

$$J_{(\mu\nu)} = \sigma_{\mu\nu} + \frac{1}{i}(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}). \tag{6.129}$$

For the translations we find

$$\varphi_r^+(\mathbf{p}) \xrightarrow{a} e^{-ipa}\varphi_r^+(\mathbf{p}), \tag{6.130}$$

$$\varphi_r^-(\mathbf{p}) \xrightarrow{a} e^{ipa}\varphi_r^-(\mathbf{p}), \tag{6.131}$$

which are the usual transformations laws, in the momentum representation, for the eigenstates of the momenta $\mathbf{p}$ and $-\mathbf{p}$ respectively.

For Lorentz transformations we find

$$\begin{aligned}\psi(x) &\xrightarrow{\Lambda} \sum_{r,r'=1}^{2}\int d\Omega_{\mathbf{p}}\left[\varphi_r^+(\mathbf{p})\mathcal{R}(\Lambda,\mathbf{p})_{r'r}u(r',\Lambda\mathbf{p})e^{-ip(\Lambda^{-1}x)} + \varphi_r^-(\mathbf{p})\mathcal{R}(\Lambda,\mathbf{p})_{r'r}v(r',\Lambda\mathbf{p})e^{ip(\Lambda^{-1}x)}\right] \\ &= \sum_{r,r'=1}^{2}\int d\Omega_{\mathbf{p}}\left[\varphi_r^+(\Lambda^{-1}\mathbf{p})\mathcal{R}(\Lambda,\Lambda^{-1}\mathbf{p})_{r'r}u(r',\mathbf{p})e^{-ipx} + \varphi_r^-(\Lambda^{-1}\mathbf{p})\mathcal{R}(\Lambda,\Lambda^{-1}\mathbf{p})_{r'r}v(r',\mathbf{p})e^{ipx}\right].\end{aligned} \tag{6.132}$$

So the law of transformation on the functions $\varphi^{\pm}$ is

$$\varphi_r^+(\mathbf{p}) \xrightarrow{\Lambda} \sum_{r'} \mathcal{R}(\Lambda, \Lambda^{-1}\mathbf{p})_{rr'} \varphi_r^+(\Lambda^{-1}\mathbf{p}), \quad (6.133)$$

$$\varphi_r^-(\mathbf{p}) \xrightarrow{\Lambda} \sum_{r'} \mathcal{R}(\Lambda, \Lambda^{-1}\mathbf{p})_{rr'} \varphi_r^-(\Lambda^{-1}\mathbf{p}). \quad (6.134)$$

This law of transformation is identical with the one constructed in Section V.A.1. The generators can be found recalling that for rotations and velocity infinitesimal transformations we have

$$\mathcal{R}(\boldsymbol{\theta}) \approx 1 + i\frac{\boldsymbol{\sigma}}{2} \cdot \boldsymbol{\theta}, \quad (6.135)$$

$$\mathcal{R}(\boldsymbol{\alpha}) \approx 1 - i\frac{\boldsymbol{\sigma}}{2} \cdot \frac{\boldsymbol{\alpha} \wedge \mathbf{p}}{p^0 + m}. \quad (6.136)$$

The result is

$$\mathbf{J} = \frac{\boldsymbol{\sigma}}{2} - i\mathbf{p} \wedge \frac{\partial}{\partial \mathbf{p}}, \quad (6.137)$$

$$\mathbf{K} = \frac{1}{2}\frac{\mathbf{p} \wedge \boldsymbol{\sigma}}{p^0 + m} + ip^0 \frac{\partial}{\partial \mathbf{p}}, \quad (6.138)$$

which coincides with the expressions (5.102) and (5.105).

Let us now write the scalar product in terms of the $\psi(x)$. Let ψ_a and ψ_b be two solutions of the Dirac equation. Then the quantity

$$J^\mu_{(a,b)}(x) = \bar{\psi}_b(x)\gamma^\mu \psi_a(x), \quad (6.139)$$

is conserved

$$\partial_\mu J^\mu_{(a,b)}(x) = 0, \quad (6.140)$$

as can easily be proved from the Dirac equation and recalling that $\gamma^0\gamma^0 = 1$ and $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$. $J^\mu_{(a,b)}$ transforms as a four-vector under Lorentz transformations

$$J^\mu_{(a,b)}(x) \xrightarrow{\Lambda} \bar{\psi}_b(\Lambda^{-1}x) S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi_a(\Lambda^{-1}x)$$
$$= (\Lambda^{-1})^\mu{}_\nu \bar{\psi}_b(\Lambda^{-1}x)\gamma^\nu \psi_a(\Lambda^{-1}x), \quad (6.141)$$

where we used Eq. (6.62) and (6.79). The conservation law is thus covariant. Applying Gauss theorem as in the scalar case, the integral extended to any space-like surface with normal $d\sigma^\mu$,

$$\int d\sigma_\mu J^\mu_{(a,b)}(x), \quad (6.142)$$

is independent from the chosen surface. Choosing a surface $x^0 =$ constant, it is independent from x^0. We thus define

$$\langle a|b\rangle = \int d\mathbf{x}\, \bar{\psi}_b(\mathbf{x}, t)\gamma^0 \psi_a(\mathbf{x}, t) = \int d\mathbf{x}\, \bar{\psi}_b(\mathbf{x}, 0)\gamma^0 \psi_a(\mathbf{x}, 0). \quad (6.143)$$

Respect to this scalar product, since it is Lorentz invariant and clearly translational invariant, the transformations of Eqs. (6.126)-(6.127) are realized as unitary operators. It can be easily shown that their generators (6.128)-(6.129) are hermitian respect to this scalar product.

Using the equations

$$u^\dagger(r, \mathbf{p})u(s, \mathbf{p}) = v^\dagger(r, \mathbf{p})v(s, \mathbf{p}) = 2p^0\delta_{rs}, \quad (6.144)$$

$$u^\dagger(r, \mathbf{p})v(s, -\mathbf{p}) = 0, \quad (6.145)$$

we obtain

$$\langle a|b\rangle = \int d\Omega_{\mathbf{p}} \left[\varphi_b^{+*}(\mathbf{p})\varphi_a^+(\mathbf{p}) + \varphi_b^{-*}(\mathbf{p})\varphi_a^-(\mathbf{p})\right]. \quad (6.146)$$

So the scalar product coincides, in the two subspaces relative to positive and negative energies, with the scalar product originally introduced for the representation of the Poincaré group.

We have then realized, in a local way, a unitary irreducible representation of the Poincaré group, extended to the parity transformations, for particles of mass m and spin 1/2.

C. Particles of spin 1

The most simple Lorentz transformation which contains spin 1 is the $(\frac{1}{2}, \frac{1}{2})$ representation, i.e the one of four-vectors. For this representation $|s_z|$ can assume the values 0 and 1.

A local wave function $W^\mu(x)$ transforms according to the law

$$W^\mu(x) \xrightarrow{\Lambda} \Lambda^\mu{}_\nu W^\nu(\Lambda^{-1}x). \tag{6.147}$$

For the state with definite momentum

$$W^\mu_{\mathbf{p}}(x) = e^{-ipx}\varepsilon^\mu(r, \mathbf{p}), \tag{6.148}$$

For the spin to be 1, in the rest frame the four-vector $\varepsilon^\mu(\mathbf{p})$ must have only spatial components. This means

$$\varepsilon^\mu(r, \mathbf{p})p_\mu = 0. \tag{6.149}$$

Then in addition to the Klein-Gordon equation

$$(\Box + m^2)W^\mu(x) = 0, \tag{6.150}$$

$W^\mu(x)$ must satisfy the constraint (6.149) which in coordinate representation translates into

$$\partial_\mu W^\mu(x) = 0. \tag{6.151}$$

The Eqs. (6.150) and (6.151) are equivalent to the system

$$G_{\mu\nu}(x) = \partial_\mu W_\nu(x) - \partial_\nu W_\mu(x), \tag{6.152}$$

$$\partial_\mu G^{\mu\nu}(x) - m^2 W^\nu(x) = 0. \tag{6.153}$$

In fact applying ∂_ν to the second equation and using the antisymmetry of $G_{\mu\nu}$ we find

$$m^2\partial_\mu W^\mu(x) = 0, \tag{6.154}$$

which coincides with Eq. (6.151) when $m \neq 0$. On the other hand if $\partial_\mu W^\mu(x) = 0$ the Eq. (6.153) coincides with (6.150).

The Eqs. (6.152) and (6.153) has both positive and negative energy solutions. The general solution is of the form

$$W^\mu(x) = \sum_{r=1}^{3} \int d\Omega_{\mathbf{p}} \left[W(r, \mathbf{p})\varepsilon^\mu(r, \mathbf{p})e^{-ipx} + \tilde{W}(r, \mathbf{p})\varepsilon^{\mu *}(r, \mathbf{p})e^{ipx}\right], \tag{6.155}$$

where $\varepsilon_\mu(r, \mathbf{p})$ are independent vectors that obey to Eq. (6.149).

By construction such solution is an irreducible representation of the Poincaré group.

We can define a scalar product, exactly in the same way we did for the spin 0 case,

$$\langle a|b\rangle = -i\int d\sigma^\nu W^*_{a\mu}(x) \overleftrightarrow{\partial}_\nu W^\mu_b(x) \tag{6.156}$$

$$= -\int d\Omega_{\mathbf{p}}\, W^*_{a\mu}(\mathbf{p})W^\mu_b(\mathbf{p}), \tag{6.157}$$

where

$$W_{a\mu}(\mathbf{p}) = \sum_{r=1}^{3} W_a(r, \mathbf{p})\varepsilon_\mu(r, \mathbf{p}). \tag{6.158}$$

Note that

$$\sum_{r=1}^{3} \varepsilon_\mu(r, \mathbf{p})\varepsilon^*_\nu(r, \mathbf{p}) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}, \tag{6.159}$$

represents the density matrix for unpolarized states. The proof is straightforward in the rest frame. The covariance fixes the form in other frames.

Let us give, for completeness, an explicit representation of the base $\varepsilon^\mu(r,\mathbf{p})$. In the rest frame we can choose any three spatial orthonormal vectors. Let them be $\boldsymbol{\varepsilon}(r,\mathbf{0})$. For particles with momentum $\mathbf{p}$ we can define, according to Eq. (5.174),

$$\varepsilon^\mu(r,\mathbf{p}) = S(\Lambda_{\mathbf{p}})\varepsilon(r,\mathbf{0}) = (\Lambda_{\mathbf{p}})^\mu{}_\nu \varepsilon^\nu(r,\mathbf{0}), \tag{6.160}$$

where we used the fact that ε^μ transform as a four-vector. Using then the explicit expression (5.80) we have

$$\begin{aligned} \varepsilon^0(r,\mathbf{p}) &= \frac{\mathbf{p}\cdot\boldsymbol{\varepsilon}(r,\mathbf{0})}{m}, & (6.161)\\ \boldsymbol{\varepsilon}(r,\mathbf{p}) &= \boldsymbol{\varepsilon}(r,\mathbf{0}) + \mathbf{p}\frac{\mathbf{p}\cdot\boldsymbol{\varepsilon}(r,\mathbf{0})}{m(p^0+m)}. & (6.162)\end{aligned}$$

The canonical base is the one where $\varepsilon^i(r,\mathbf{0}) = \delta_{ir}$. Choosing instead as a base the eigenstates of s_z we have

$$\begin{aligned} \boldsymbol{\varepsilon}(+1,\mathbf{0}) &= -\frac{i}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y), & (6.163)\\ \boldsymbol{\varepsilon}(0,\mathbf{0}) &= i\mathbf{e}_z, & (6.164)\\ \boldsymbol{\varepsilon}(-1,\mathbf{0}) &= \frac{i}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y), & (6.165)\end{aligned}$$

where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are the versors of the axes.

In the vectorial case the Wigner matrix $\mathcal{R}$ is defined by

$$\mathcal{R}(\Lambda)_{r'r}\varepsilon^\mu(r',\mathbf{p}) = \Lambda^\mu{}_\nu \varepsilon^\nu(r,\Lambda^{-1}\mathbf{p}). \tag{6.166}$$

VII. THE SECOND QUANTIZATION

It is an experimental fact that the number of particles may change in physical processes: An hydrogen atom in the state $2P$ is composed by an electron and a proton and decays into an atom in its fundamental state plus a photon, an electron which pass through the Coulomb field of nucleus is accelerated and emit photons (Bremsstrahlung), when a positron annihilates with an electron their mass is converted in energy in the form of two photons, in the scattering between two high energy protons many pions are produced, ... Then, exist transitions between states with different number of particles. In Section VII.A we will present a formalism that allows to describe systems of many free particles, used in any many-body theory, relativistic or not, and known as Fock method. It allows to describe many particles states with the correct statistics and to introduce operators that change the number of particles (creation and annihilation operators).

In Section VII.B we will introduce the free field operators, and we will interpret in terms of field operators the negative energy solutions of the equations of free motion.

The relativistic equations of motion can be rederived in the Lagrangian formalism and it can be shown that the Fock second quantization is equivalent to the canonical quantization of a system of an infinite number of degrees of freedom.

The Lagrangian formalism is indispensable to write theories of non-free particles: In interaction.

A. Fock space

Let us consider an orthonormal complete base $|i\rangle$ for the single particle states. For example the base $|r,\mathbf{p}\rangle$ of the positive energy states for relativistic particles introduced in Section VI.

If the particles are bosons, in the state $|i\rangle$ can coexist an arbitrary number n_i of free particles.

If the particles are fermions, in the state $|i\rangle$ can exist at most one particle.

In both cases, assigning the occupation numbers $\{n_i\}$ in the various states $|i\rangle$ determines completely the state of the system, since the state must be symmetric for the bosons and completely antisymmetric for the fermions.

1. Bosons

For any state $|i\rangle$ the observable number of particles in such state, n_i, has integer eigenvalues: $1, 2, 3, \ldots$

His spectrum is the one of an harmonic oscillator. As for the harmonic oscillator is possible to define a rising (creation) operator $b_i^\dagger$ and a lowering (annihilation) operator b_i of the eigenvalue of n_i. The commutation properties are

$$[b_i, b_i^\dagger] = 1 \quad [b_i, b_i] = [b_i^\dagger, b_i^\dagger] = 0, \tag{7.1}$$

We then define $n_i = b_i^\dagger b_i$ with

$$[n_i, b_i] = -b_i \quad [n_i, b_i^\dagger] = b_i^\dagger. \tag{7.2}$$

The lower state $|0_i\rangle$ corresponds to zero particles in the state $|i\rangle$ and $b_i|0_i\rangle = 0$ with $\langle 0_i|0_i\rangle = 1$. The normalized state with n_i particles is then

$$\frac{(b_i^\dagger)^{n_i}}{\sqrt{n_i!}}|0_i\rangle = |n_i\rangle. \tag{7.3}$$

A state identified by the set of occupation numbers $\{n_i\}$ in the different states $|i\rangle$ can be written as

$$|n_{i_1}, \ldots, n_{i_k}, \ldots\rangle = \prod_{i_i} \frac{(b_i^\dagger)^{n_i}}{\sqrt{n_i!}}|0\rangle \tag{7.4}$$

where $|0\rangle = \prod_i |0_i\rangle$ is the vacuum. It is automatically symmetric under particle exchange if

$$[b_i, b_k] = [b_i^\dagger, b_k^\dagger] = 0. \tag{7.5}$$

The "harmonic oscillators" correspondent to different modes are independent and we must also have

$$[b_i, b_k^\dagger] = \delta_{ik}. \tag{7.6}$$

The total number of particles is

$$N = \sum_i n_i = \sum_i b_i^\dagger b_i, \tag{7.7}$$

Moreover $\langle 0|0\rangle = 1$.

2. Fermions

For the fermions the occupation number can be 0 or 1 and the state must be completely antisymmetric under particle exchange. This can be realized by associating to each single particle state an harmonic anti-oscillator, requiring anticommutation between operators relative to different modes

$$[b_i, b_k]_+ = [b_i^\dagger, b_k^\dagger]_+ = 0 \quad [b_i, b_k^\dagger]_+ = \delta_{ik}, \tag{7.8}$$

$$n_i = b_i^\dagger b_i \quad N = \sum_i n_i \quad b_i|0_i\rangle = 0, \tag{7.9}$$

$$[n_i, b_k] = -b_i\delta_{ik} \quad [n_i, b_k^\dagger] = b_i^\dagger \delta_{ik}. \tag{7.10}$$

The subscript $+$ indicates the anticommutator. The possible states in the mode $|i\rangle$ are $|0_i\rangle$ and $b_i^\dagger|0_i\rangle = |1_i\rangle$. ${b_i^\dagger}^2|0_i\rangle = 0$ because the operator $b_i^\dagger$ anticommutes with itself. Moreover

$$b_i b_i^\dagger|0_i\rangle = (-b_i b_i^\dagger + 1)|0_i\rangle = |0_i\rangle. \tag{7.11}$$

Given an operator O written in terms of creation and annihilation operators we will denote with $:O:$ the normal ordered operator for bosons or the antinormal ordered operator for fermions. For bosons it is obtained from O displacing all creation operators to the left and all annihilation operators to the right and for fermions is is obtained from O displacing all creation operators to the left and all annihilation operators to the right times $(-1)^n$, with n the number of needed exchanges of a creation and an annihilation operator. For example for bosons $:bb^\dagger: = b^\dagger b = bb^\dagger - 1$. Normal ordering is not linear. For example $:bb^\dagger: =: 1 + b^\dagger b: =: 1: +: b^\dagger b: = 1 + b^\dagger b \neq b^\dagger b$. For fermions $:bb^\dagger: = -b^\dagger b = bb^\dagger - 1$. In particular we will always have $\langle 0|: O: 0\rangle = 0$ on the vacuum. We usually refer to the normal order as the Wick order.

The (anti)commutation relations are invariant under unitary changes of base. Let V be a unitary transformation from the base $|1_i\rangle$ for the single particle states to the base $|1_\alpha\rangle$

$$|1_\alpha\rangle = \sum_i V_{\alpha i}|1_i\rangle \quad |1_i\rangle = \sum_i V^\dagger_{i\alpha}|1_\alpha\rangle, \tag{7.12}$$

with $VV^\dagger = V^\dagger V = 1$. If $|1_i\rangle = b_i^\dagger|0\rangle$ then $|1_\alpha\rangle = \sum_i V_{\alpha i} b_i^\dagger |0\rangle$. Defining

$$b^\dagger_\alpha = \sum_i V_{\alpha i} b_i^\dagger \quad b_\alpha = \sum_i V^*_{\alpha i} b_i, \tag{7.13}$$

we have

$$[b_\alpha, b_\beta]_\pm = [b^\dagger_\alpha, b^\dagger_\beta]_\pm = 0, \tag{7.14}$$

$$[b_\alpha, b^\dagger_\beta]_\pm = \sum_{ij} V^*_{\alpha i} V_{\beta j}[b_i, b^\dagger_j]_\pm = \sum_i V^*_{\alpha i} V_{\beta i} = (VV^\dagger)_{\beta\alpha} = \delta_{\alpha\beta}. \tag{7.15}$$

The vacuum remains unchanged.

If the index i that label the states is continuous, as for the momentum $\mathbf{p}$ in the base $|r, \mathbf{p}\rangle$ for free particles, the (anti)commutation rules must be modified replacing the δ_{ik} in the Eqs. (7.6) and (7.8) the diagonal element of the identity matrix in the chosen representation. For the states $|r, \mathbf{p}\rangle$

$$[b(r, \mathbf{p}), b(r', \mathbf{p}')]_\pm = [b^\dagger(r, \mathbf{p}), b^\dagger(r', \mathbf{p}')]_\pm = 0, \tag{7.16}$$

$$[b(r, \mathbf{p}), b^\dagger(r', \mathbf{p}')]_\pm = \delta_{rr'}(2\pi)^3 2p^0 \delta(\mathbf{p} - \mathbf{p}'), \tag{7.17}$$

where $\pm$ denotes the commutator or anticommutator. This choice give the correct states normalization. In fact

$$\langle r, \mathbf{p}|r', \mathbf{p}'\rangle = \langle 0|b(r, \mathbf{p})b^\dagger(r', \mathbf{p}')0\rangle = \langle 0|[b(r, \mathbf{p}), b^\dagger(r', \mathbf{p}')]_\pm 0\rangle = \delta_{rr'}(2\pi)^3 2p^0\delta(\mathbf{p} - \mathbf{p}'). \tag{7.18}$$

The density of occupation number is $b^\dagger(r, \mathbf{p})b(r, \mathbf{p})$ and the total number of particles is

$$N = \int d\Omega_{\mathbf{p}} \sum_r b^\dagger(r, \mathbf{p})b(r, \mathbf{p}). \tag{7.19}$$

The commutation rules for N are

$$[N, b(r, \mathbf{p})] = -b(r, \mathbf{p}) \quad [N, b^\dagger(r, \mathbf{p})] = b^\dagger(r, \mathbf{p}). \tag{7.20}$$

B. Field operators

Let

$$|s\rangle = \int d\Omega_{\mathbf{p}} \sum_r \varphi_s(r, \mathbf{p})|r, \mathbf{p}\rangle, \tag{7.21}$$

be any single particle state. It can be written as

$$|s\rangle = \int d\Omega_{\mathbf{p}} \sum_r \varphi_s(r, \mathbf{p}) b^\dagger(r, \mathbf{p})|0\rangle, \tag{7.22}$$

with

$$\langle r', \mathbf{p}'|s\rangle = \langle 0|b(r', \mathbf{p}')s\rangle = \int d\Omega_{\mathbf{p}} \sum_r \varphi_s(r, \mathbf{p})\langle 0|b(r', \mathbf{p}')b^\dagger(r, \mathbf{p})0\rangle, \tag{7.23}$$

but

$$\langle 0|b(r', \mathbf{p}')b^\dagger(r, \mathbf{p})0\rangle = \delta_{rr'}(2\pi)^3 2p^0\delta(\mathbf{p} - \mathbf{p}'), \tag{7.24}$$

and so

$$\langle 0|b(r', \mathbf{p}')s\rangle = \varphi_s(r', \mathbf{p}'). \tag{7.25}$$

The operator $b(r, \mathbf{p})$ extracts from a state the component with momentum $\mathbf{p}$. We can construct an operator which acts in the same way on the x space. For a particle of any spin let us consider the positive energy solutions and build the following operator

$$\varphi_+(x) = \int d\Omega_{\mathbf{p}} \sum_r b(r, \mathbf{p})u(r, \mathbf{p})e^{-ipx}. \tag{7.26}$$

The operator $\varphi_+(x)$ has the same number of components of the function $u(r, \mathbf{p})$: 1 for spin 0, 4 for spin 1/2 and 1. In any case from Eq. (7.26) follows

$$\langle 0|\varphi_+(x)s\rangle = \int d\Omega_{\mathbf{p}} \sum_r \varphi_s(r, \mathbf{p})u(r, \mathbf{p})e^{-ipx} = \varphi_s(x), \tag{7.27}$$

where $\varphi_s(x)$ is the wave function of the state $|s\rangle$.

The operator $\varphi_+(x)$ defined in Eq. (7.26) is called field operator or better the positive energy component of the field operator. The subscript + indicates that it contains only positive energies.

The operator $\varphi_+(x)$ is a linear superposition of solutions $u(r, \mathbf{p})e^{-ipx}$ with positive energy of the wave equation, so it is a solution with positive energy of the wave equation.

Let us give the explicit formulas for the field operator

$$\text{spin } 0 \quad \varphi_+(x) = \int d\Omega_{\mathbf{p}}\, b(\mathbf{p})e^{-ipx}, \tag{7.28}$$

$$\text{spin } \tfrac{1}{2} \quad \psi_+(x) = \int d\Omega_{\mathbf{p}} \sum_{r=1}^{2} u(r, \mathbf{p})b(r, \mathbf{p})e^{-ipx}, \tag{7.29}$$

$$\text{spin } 1 \quad W_+^\mu(x) = \int d\Omega_{\mathbf{p}} \sum_{r=1}^{3} \varepsilon^\mu(r, \mathbf{p})b(r, \mathbf{p})e^{-ipx}. \tag{7.30}$$

It is possible to invert these formulas using the expressions for the scalar products defined in the various cases (6.22), (6.143), and (6.156)

$$\text{spin } 0 \quad b(\mathbf{p}) = i\int d\sigma^\mu\, e^{ipx} \overleftrightarrow{\partial}_\mu\, \varphi_+(x) = i\int d\mathbf{x}\, e^{ipx} \overleftrightarrow{\partial}_0\, \varphi_+(x), \tag{7.31}$$

$$\text{spin } \tfrac{1}{2} \quad b(r, \mathbf{p}) = \int d\mathbf{x}\, u^\dagger(r, \mathbf{p})e^{ipx}\psi_+(x), \tag{7.32}$$

$$\text{spin } 1 \quad b(r, \mathbf{p}) = -i\int d\mathbf{x}\, \varepsilon_\mu^*(r, \mathbf{p})e^{ipx} \overleftrightarrow{\partial}_0\, W_+^\mu(x). \tag{7.33}$$

All observables can be expressed in terms of $b^\dagger(r, \mathbf{p})$ and $b(r, \mathbf{p})$. Then they can be expressed in terms of the fields and of their first derivatives for spin 0 and 1 particles, and in terms of the fields for spin 1/2 particles.

C. Transformation properties of the field operators

The invariance under a symmetry group implies the existence of a unitary representation of the group which send the Hilbert space into itself.

For a free particle the symmetry group is the Poincaré group and the representation is irreducible. We want now construct the representation of the group on the many free particles states.

Let $U(\Lambda, a) = T_a U(\Lambda)$ be a transformation of the group with Lorentz matrix Λ and translation parameter a^μ. On the single particle states we know that

$$U(\Lambda, a)|r, \mathbf{p}\rangle = e^{-i(\Lambda p)a} \mathcal{R}(\Lambda, \mathbf{p})_{r'r} |r', \Lambda \mathbf{p}\rangle, \tag{7.34}$$

where $\mathcal{R}$ is a unitary matrix which represents the Wigner rotation. To construct the representation of the group in the Fock space we assume that the vacuum is invariant

$$U(\Lambda, a)|0\rangle = |0\rangle, \tag{7.35}$$

and we set

$$U(\Lambda, a) b^\dagger(r, \mathbf{p}) U^\dagger(\Lambda, a) = e^{-i(\Lambda p)a} \mathcal{R}(\Lambda, \mathbf{p})_{r'r} b^\dagger(r', \Lambda \mathbf{p}). \tag{7.36}$$

This representation realizes the (7.34) and transforms independently the many particles states. For the annihilation operator we will then have

$$U(\Lambda, a) b(r, \mathbf{p}) U^\dagger(\Lambda, a) = e^{i(\Lambda p)a} \mathcal{R}(\Lambda, \mathbf{p})^*_{rr'} b(r', \Lambda \mathbf{p}). \tag{7.37}$$

We define the transformed of $b(r, \mathbf{p})$ as follows [4]

$$b(r, \mathbf{p}) \to U^\dagger(\Lambda, a) b(r, \mathbf{p}) U(\Lambda, a). \tag{7.39}$$

From Eq. (7.37), recalling that

$$U^{-1}(\Lambda, a) = U(\Lambda^{-1}, -\Lambda^{-1} a), \tag{7.40}$$

we find

$$U^\dagger(\Lambda, a) b(r, \mathbf{p}) U(\Lambda, a) = e^{-ipa} \mathcal{R}(\Lambda, \Lambda^{-1}\mathbf{p})_{rr'} b(r', \Lambda^{-1}\mathbf{p}), \tag{7.41}$$

$$U^\dagger(\Lambda, a) b^\dagger(r, \mathbf{p}) U(\Lambda, a) = e^{ipa} \mathcal{R}(\Lambda, \Lambda^{-1}\mathbf{p})^*_{rr'} b(r', \Lambda^{-1}\mathbf{p}). \tag{7.42}$$

(7.43)

To derive Eq. (7.41) we used

$$\mathcal{R}(\Lambda^{-1}, \mathbf{p})^*_{r'r} = \mathcal{R}(\Lambda^{-1}, \mathbf{p})^\dagger_{rr'}, \tag{7.44}$$

and

$$\mathcal{R}(\Lambda^{-1}, \mathbf{p})^\dagger_{rr'} = \mathcal{R}(\Lambda, \Lambda^{-1}\mathbf{p})_{rr'}. \tag{7.45}$$

Eq. (7.45) can be derived observing that $\mathcal{R}$ is unitary, that

$$|r, \mathbf{p}\rangle = U(\Lambda) U^\dagger(\Lambda) |r, \mathbf{p}\rangle = U(\Lambda) \mathcal{R}(\Lambda^{-1}, \mathbf{p})_{r'r} |r', \Lambda^{-1}\mathbf{p}\rangle \tag{7.46}$$

$$= \mathcal{R}(\Lambda, \Lambda^{-1}\mathbf{p})_{r''r'} \mathcal{R}(\Lambda^{-1}, \mathbf{p})_{r'r} |r'', \mathbf{p}\rangle, \tag{7.47}$$

and that $|r, \mathbf{p}\rangle$ is a complete base at fixed $\mathbf{p}$. Since the transformation (7.41) is unitary in Fock space it leaves unchanged the commutation relations.

The generators of the unitary transformation $U(\Lambda, a)$ can be explicitly constructed as hermitian operators on Fock space. For infinitesimal transformations

$$U(\Lambda, a) \approx 1 - i p_\mu a^\mu + i \boldsymbol{\theta} \cdot \mathbf{J} - i \boldsymbol{\alpha} \cdot \mathbf{K}. \tag{7.48}$$

---

[4] Note that here we must define the transformed operator using the inverse transformation respect to the one that applies to regular observables for which the measure in the two reference frames must coincide. In fact

$$\varphi_s(r, \mathbf{p}) = \langle 0|bs\rangle \to \langle 0|U^\dagger b U s\rangle \equiv \langle 0|b' s\rangle, \tag{7.38}$$

where b' is the transformed operator and in the last equation we used the fact that $U|0\rangle = |0\rangle$ and $U|s\rangle = |s'\rangle$.

We recall that for infinitesimal rotations

$$\mathcal{R}(\Lambda)_{rr'} \approx \delta_{rr'} + i\boldsymbol{\theta}\cdot\mathbf{s}_{rr'}, \tag{7.49}$$

$$b(r,\Lambda^{-1}\mathbf{p}) \approx b(r,\mathbf{p}+\boldsymbol{\theta}\wedge\mathbf{p}) \approx b(r,\mathbf{p}) + \boldsymbol{\theta}\cdot\left(\mathbf{p}\wedge\frac{\partial}{\partial\mathbf{p}}\right)b(r,\mathbf{p}), \tag{7.50}$$

and for infinitesimal velocity transformations

$$\mathcal{R}(\Lambda)_{rr'} \approx \delta_{rr'} - i\frac{\boldsymbol{\alpha}\wedge\mathbf{p}}{p^0+m}\cdot\mathbf{s}_{rr'}, \tag{7.51}$$

$$b(r,\Lambda^{-1}\mathbf{p}) \approx b(r,\mathbf{p}+\boldsymbol{\alpha}p^0) \approx b(r,\mathbf{p}) + \boldsymbol{\alpha}\cdot p^0\frac{\partial}{\partial\mathbf{p}}b(r,\mathbf{p}). \tag{7.52}$$

Using Eqs. (7.41) and (7.48) we derive the commutation relations for the generators

$$[p_\mu, b(r,\mathbf{p})] = -p_\mu b(r,\mathbf{p}), \tag{7.53}$$

$$[\mathbf{J}, b(r,\mathbf{p})] = -\left(\mathbf{s} - i\mathbf{p}\wedge\frac{\partial}{\partial\mathbf{p}}\right)_{rr'} b(r',\mathbf{p}), \tag{7.54}$$

$$[\mathbf{K}, b(r,\mathbf{p})] = -\left(\frac{\mathbf{p}\wedge\mathbf{s}}{p^0+m} + ip^0\frac{\partial}{\partial\mathbf{p}}\right)_{rr'} b(r',\mathbf{p}). \tag{7.55}$$

Taking the hermitian conjugate and recalling that the $\mathbf{s}$ matrices are hermitian we find

$$[p_\mu, b^\dagger(r,\mathbf{p})] = p_\mu b^\dagger(r,\mathbf{p}), \tag{7.56}$$

$$[\mathbf{J}, b^\dagger(r,\mathbf{p})] = \left(\mathbf{s} + i\mathbf{p}\wedge\frac{\partial}{\partial\mathbf{p}}\right)_{r'r} b^\dagger(r',\mathbf{p}), \tag{7.57}$$

$$[\mathbf{K}, b^\dagger(r,\mathbf{p})] = \left(\frac{\mathbf{p}\wedge\mathbf{s}}{p^0+m} - ip^0\frac{\partial}{\partial\mathbf{p}}\right)_{r'r} b^\dagger(r',\mathbf{p}). \tag{7.58}$$

It is possible to give an explicit representation for the operators p_μ, $\mathbf{J}$, and $\mathbf{K}$ in terms of the operators b and $b^\dagger$

$$p_\mu = \int d\Omega_{\mathbf{p}} \sum_r b^\dagger(r,\mathbf{p}) p_\mu b(r,\mathbf{p}), \tag{7.59}$$

$$\mathbf{J} = \int d\Omega_{\mathbf{p}} \sum_r b^\dagger(r,\mathbf{p})\left(\mathbf{s} - i\mathbf{p}\wedge\frac{\partial}{\partial\mathbf{p}}\right)_{rr'} b(r,\mathbf{p}), \tag{7.60}$$

$$\mathbf{K} = \int d\Omega_{\mathbf{p}} \sum_r b^\dagger(r,\mathbf{p})\left(\frac{\mathbf{p}\wedge\mathbf{s}}{p^0+m} + ip^0\frac{\partial}{\partial\mathbf{p}}\right)_{rr'} b(r,\mathbf{p}), \tag{7.61}$$

so that these operators satisfy the commutation rules (7.53)-(7.55).

Let us now treat the transformation properties of the field operator. The Eq. (7.41) induces the following transformation

$$U^\dagger(\Lambda,a)\varphi_+(x)U(\Lambda,a) = \int d\Omega_{\mathbf{p}}\, e^{-ipx}\sum_r u(r,\mathbf{p})e^{-ipa}\mathcal{R}(\Lambda,\Lambda^{-1}\mathbf{p})_{rr'}b(r',\Lambda^{-1}\mathbf{p}). \tag{7.62}$$

Changing variables $\mathbf{p}\to\Lambda\mathbf{p}$ and using Eq. (6.122) we find

$$\varphi'(x) \equiv U^\dagger(\Lambda,a)\varphi_+(x)U(\Lambda,a) = S(\Lambda)\varphi_+(\Lambda^{-1}x + \Lambda^{-1}a), \tag{7.63}$$

which is the correct transformation law for a local operator [5]. Indicating with x' the transformed event we can also write

$$\varphi'_+(x') = U^\dagger\varphi_+(x')U = S(\Lambda)\varphi_+(x). \tag{7.64}$$

---

[5] We recall that $(\Lambda,a)^{-1}x = (T_a\Lambda)^{-1}x = \Lambda^{-1}T_{-a}x = \Lambda^{-1}x + \Lambda^{-1}a$.

This equation allows to write down immediately the action of the generators of the Poincaré group on the field operators. Denoting with $J_{(\mu\nu)}$ and p_μ the generators in the Fock space

$$U(T_a) = e^{-ia^\mu p_\mu} \quad U(\Lambda) = e^{\frac{i}{2}\omega^{\mu\nu}J_{(\mu\nu)}}, \tag{7.65}$$

and with $\sigma_{\mu\nu}$ the generator of the group in the representation under which φ transforms, i.e. the generator of the $S(\Lambda)$ matrix, from Eq. (7.63) follows

$$[p_\mu, \varphi_+(x)] = -i\partial_\mu \varphi_+(x), \tag{7.66}$$

$$[J_{(\mu\nu)}, \varphi_+(x)] = -\left[\sigma_{\mu\nu} - i(x_\mu\partial_\nu - x_\nu\partial_\mu)\right]\varphi_+(x), \tag{7.67}$$

as follows from Eqs. (6.128) and (6.129).

D. Locality and spin-statistics theorem

In constructing the relativistic theory it is necessary to deal with local operators commuting at spacelike distances. In fact, since a signal can not propagate at speeds higher than that of light, measures occurred at spatial distances must not influence each other. As observed in Section VII.B all observables can be written in terms of fields and their first derivatives. If the (anti)commutators between these quantities are zero for spacelike distances it will be possible to construct a theory that satisfies causality.

From the commutators between the operators $b(r, \mathbf{p})$ and $b^\dagger(r, \mathbf{p})$ we can easily calculate the commutators between the fields and their derivatives. Let us consider first the scalar field

$$[\varphi_+(x), \varphi_+(y)] = 0, \tag{7.68}$$

$$[\varphi_+(x), \varphi_+^\dagger(y)] = F_+(x-y), \tag{7.69}$$

$$[\varphi_+(x), \partial_0\varphi_+^\dagger(y)] = \frac{\partial}{\partial y^0}F_+(x-y), \tag{7.70}$$

where Eq. (7.70) follows from Eq. (7.69).

The function F_+ is invariant under translations and under Lorentz transformations. It is in fact a c-number, i.e. as an operator in the Fock space it is proportional to the identity, because such is $[b(r, \mathbf{p}), b^\dagger(r, \mathbf{p})]$. From Eq. (7.69) follows that

$$U^\dagger(\Lambda, a)[\varphi_+(x), \varphi_+^\dagger(x)]U(\Lambda, a) = F_+(x-y)U^\dagger(\Lambda, a)U(\Lambda, a) = F_+(x-y). \tag{7.71}$$

But the first member is also equal to

$$[\varphi_+(\Lambda^{-1}(x+a)), \varphi_+^\dagger(\Lambda^{-1}(x+a))] = F_+(\Lambda^{-1}(x-y)), \tag{7.72}$$

and this proves the invariance of F_+ under the Poincaré group.

Explicitly we have

$$F_+(x-y) = \int d\Omega_{\mathbf{p}}\, e^{-ip(x-y)}. \tag{7.73}$$

If x and y are at spacelike distances it is always possible to bring them to be simultaneous ($x^0 = y^0$) through a Lorentz transformation. To study the behavior of F_+ at spacelike distances it is sufficient to study it at equal times ($x^0 = y^0$). We then have

$$F_+(0, \mathbf{x}-\mathbf{y}) = \int d\Omega_{\mathbf{p}}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}, \tag{7.74}$$

$$\left.\frac{\partial}{\partial y^0}F_+(x^0-y^0, \mathbf{x}-\mathbf{y})\right|_{y^0=x^0} = \frac{i}{2}\int \frac{d\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \frac{i}{2}\delta(\mathbf{x}-\mathbf{y}). \tag{7.75}$$

The integral in Eq. (7.74) can be easily calculated in terms of Bessel functions

$$F_+(0, \mathbf{x}-\mathbf{y}) = \frac{m}{(2\pi)^2|\mathbf{x}-\mathbf{y}|}K_0(m|\mathbf{x}-\mathbf{y}|). \tag{7.76}$$

F_+ is different from zero at spacelike distances of the order of the Compton wavelength of the particle ($\ell = h/mc$). So a theory constructed in terms of just the φ_+ is non local.

But we remember that next to the positive energy solutions exist the "negative energy" solutions of the Klein-Gordon equation. In the Fock space context a dependence of the kind e^{ipx} is associated to a creation operator, rather than to a destruction operator as in the expansion for φ_+. While considering the negative energy solutions is then natural to introduce a "negative frequency" field

$$\varphi_-(x) = \int d\Omega_{\mathbf{p}}\, e^{ipx} d^\dagger(\mathbf{p}). \tag{7.77}$$

The operators $d^\dagger(\mathbf{p})$ and $d(\mathbf{p})$ are operator independent from $b^\dagger(\mathbf{p})$ and $b(\mathbf{p})$, i.e. they describe a different particle, and so they commute with them.

Let us now construct the field

$$\varphi(x) = \varphi_+(x) + \varphi_-(x), \tag{7.78}$$

or

$$\begin{aligned} \varphi(x) &= \int d\Omega_{\mathbf{p}} \left[d(\mathbf{p})e^{-ipx} + d^\dagger(\mathbf{p})e^{ipx} \right], & (7.79)\\ \varphi^\dagger(x) &= \int d\Omega_{\mathbf{p}} \left[d^\dagger(\mathbf{p})e^{ipx} + d(\mathbf{p})e^{-ipx} \right]. & (7.80)\\ & & (7.81) \end{aligned}$$

The commutators now becomes

$$\begin{aligned} [\varphi(x), \varphi(y)] &= [\varphi^\dagger(x), \varphi^\dagger(y)] = 0, & (7.82)\\ [\varphi(x), \varphi^\dagger(y)] &= F_+(x-y) - F_+(y-x), & (7.83)\\ [\varphi(x), \partial_0\varphi^\dagger(y)] &= \frac{\partial}{\partial y^0}\left[F_+(x-y) - F_+(y-x)\right]. & (7.84) \end{aligned}$$

At equal times, at spacelike distances, we have

$$\begin{aligned} [\varphi(x^0,\mathbf{x}), \varphi^\dagger(x^0,\mathbf{y})] &= 0, & (7.85)\\ [\varphi(x^0,\mathbf{x}), \partial_0\varphi^\dagger(x^0,\mathbf{y})] &= i\delta(\mathbf{x}-\mathbf{y}). & (7.86) \end{aligned}$$

The theory is now *local*.

We note that the minus sign in the Eqs. (7.83) and (7.84) depends by the choice of commutation relation: The locality in Eqs. (7.85) and (7.86) would have been destroyed if we would have chosen the Fermi statistics. This is a manifestation of the so called *spin-statistics theorem*.

We note that since $\varphi(x)$ is a superposition of solutions of the Klein-Gordon equation it itself satisfies to such equation

$$(\Box + m^2)\varphi(x) = 0. \tag{7.87}$$

Note that since $b(\mathbf{p}) \neq d(\mathbf{p})$ the scalar field is not hermitian. This is also called a *charged* scalar field. The hermitian field is called *neutral*. The particle described by the creation operator $d^\dagger$ is called *antiparticle*.

Let us now treat the spin 1/2 case. For the Dirac field,

$$\psi_+(x) = \int d\Omega_{\mathbf{p}} \sum_r u(r,\mathbf{p})b(r,\mathbf{p})e^{-ipx}, \tag{7.88}$$

we have

$$\begin{aligned} [\psi_+^\alpha(x), {\psi_+^\dagger}^\beta(y)]_+ &= \int d\Omega_{\mathbf{p}} \sum_r u^\alpha(r,\mathbf{p}){u^\dagger}^\beta(r,\mathbf{p})e^{-ip(x-y)}\\ &= \int d\Omega_{\mathbf{p}} \left[(\not{p}+m)\gamma^0\right]^{\alpha\beta} e^{-ip(x-y)}, & (7.89) \end{aligned}$$

where we used the anticommutation relations for the $b, b^\dagger$ and we used the Eq. (6.109) for the projector on the positive energies states.

Omitting the indexes α, β and using the anticommutation rules of the γ matrices we can then write

$$[\psi_+(x), \psi_+^\dagger(y)]_+ = \left(i\frac{\partial}{\partial x^0} + m\gamma^0 + i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\right)F_+(x-y), \tag{7.90}$$

where F_+ is again given by Eq. (7.74). At equal times

$$[\psi_+(x^0,\mathbf{x}), \psi_+^\dagger(x^0,\mathbf{y})]_+ = \frac{i}{2}\delta(\mathbf{x}-\mathbf{y}) + (m\gamma^0 + i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla})F_+(x-y), \tag{7.91}$$

which is non-local.

In analogy to what we did in the scalar case we introduce

$$\psi_-(x) = \int d\Omega_{\mathbf{p}} \sum_r v(r,\mathbf{p})b^\dagger(r,\mathbf{p})e^{ipx}, \tag{7.92}$$

where $d^\dagger$ is the creation operator for a new particle

$$\begin{aligned}
[d(r,\mathbf{p}), d^\dagger(r',\mathbf{p}')]_+ &= \delta_{rr'}2p^0(2\pi)^3\delta(\mathbf{p}-\mathbf{p}'), &(7.93)\\
[d,d]_+ &= [d^\dagger,d^\dagger]_+ = [b,d]_+ = [b,d^\dagger]_+ = [b^\dagger,d]_+ = [b^\dagger,d^\dagger]_+0, &(7.94)
\end{aligned}$$

and

$$\begin{aligned}
\psi(x) = \psi_+(x) + \psi_-(x) &= \int d\Omega_{\mathbf{p}}\sum_r \left[u(r,\mathbf{p})b(r,\mathbf{p})e^{-ipx} + v(r,\mathbf{p})d^\dagger(r,\mathbf{p})e^{ipx}\right], &(7.95)\\
\psi^\dagger(x) &= \int d\Omega_{\mathbf{p}}\sum_r \left[u^\dagger(r,\mathbf{p})b^\dagger(r,\mathbf{p})e^{ipx} + v^\dagger(r,\mathbf{p})d(r,\mathbf{p})e^{-ipx}\right]. &(7.96)
\end{aligned}$$

Then

$$\begin{aligned}
[\psi(x),\psi(y)]_+ &= [\psi^\dagger(x),\psi^\dagger(y)]_+ = 0, &(7.97)\\
[\psi(x),\psi^\dagger(y)]_+ &= \int d\Omega_{\mathbf{p}}\left[(\not p + m)\gamma^0 e^{-ip(x-y)} + (\not p - m)\gamma^0 e^{ip(x-y)}\right] &(7.98)\\
&= \int d\Omega_{\mathbf{p}}\left[\left(i\gamma^\mu\frac{\partial}{\partial x^\mu} + m\right)\gamma^0 e^{-ip(x-y)} + \left(i\gamma^\mu\frac{\partial}{\partial y^\mu} - m\right)\gamma^0 e^{ip(x-y)}\right]\\
&= \left(i\gamma^\mu\frac{\partial}{\partial x^\mu} + m\right)\gamma^0[F_+(x-y) - F_+(y-x)].
\end{aligned}$$

At equal times, using $\gamma^0\gamma^0 = 1$, we find

$$[\psi(x^0,\mathbf{x}), \psi^\dagger(x^0,\mathbf{y})]_+ = i\delta(\mathbf{x}-\mathbf{y}), \tag{7.99}$$

which is again local. Again we must notice that in order to have Eq. (7.99) in a local form it was essential to choose the anticommutators. The commutator would have brought a minus sign for the $vv^\dagger$ term in Eq. (7.98) and to a non-local result. This is a manifestation of the spin-statistic theorem.

Since ψ is a linear superposition of Dirac equation solutions, it itself is a solution of the Dirac equation

$$(i\not\partial - m)\psi(x) = 0. \tag{7.100}$$

Let us conclude with the case of a massive vectorial field. The analysis is identical to the scalar case. For a vectorial field we define

$$\begin{aligned}
W_\mu(x) &= \int d\Omega_{\mathbf{p}}\sum_{r=1}^{3}[\varepsilon_\mu(r,\mathbf{p})b(r,\mathbf{p})e^{-ipx} + \varepsilon_\mu^*(r,\mathbf{p})d^\dagger(r,\mathbf{p})e^{ipx}], &(7.101)\\
W_\mu^\dagger(x) &= \int d\Omega_{\mathbf{p}}\sum_{r=1}^{3}[\varepsilon_\mu^*(r,\mathbf{p})b^\dagger(r,\mathbf{p})e^{ipx} + \varepsilon_\mu(r,\mathbf{p})d(r,\mathbf{p})e^{-ipx}]. &(7.102)
\end{aligned}$$

The commutation rules can be easily derived recalling Eq. (6.159)

$$[W_\mu(x), W_\nu^\dagger(y)] = -\left(g_{\mu\nu} + \frac{1}{m^2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right)[F_+(x-y) - F_+(y-x)], \tag{7.103}$$

$$[W_\mu(x^0, \mathbf{x}), W_\nu^\dagger(x^0, \mathbf{y})] = -\frac{i}{2m^2}[g_{\mu 0}\partial_\nu + g_{0\nu}\partial_\mu]\delta(\mathbf{x}-\mathbf{y}), \tag{7.104}$$

$$[W_\mu(x^0, \mathbf{x}), \partial_0 W_\nu^\dagger(x^0, \mathbf{y})] = -\left(g_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{m^2}\right) i\delta(\mathbf{x}-\mathbf{y}). \tag{7.105}$$

Also in this case the use of the Bose statistics has been essential for the locality of (7.104). Again this is a manifestation of the spin-statistics theorem.

The vectorial field W_μ will satisfy to the following system of equations

$$(\Box + m^2)W^\mu(x) = 0, \tag{7.106}$$

$$\partial_\mu W^\mu = 0 \tag{7.107}$$

The spin-statistics theorem states that, as a consequence of Lorentz invariance and of locality, half integer spin particles must obey to Fermi statistics and integer spin particles must obey to Bose statistics.

As we saw in the various cases, the introduction of the negative energy solutions does not interfere with the Lorentz structure of the fields. Since the commutation rules of the operators b and d are identical we can write the action of the group on the whole Fock space generated by $b^\dagger$ and $d^\dagger$. In particular the generators are given by

$$p_\mu = \int d\Omega_{\mathbf{p}} \sum_r \left[b^\dagger(r,\mathbf{p})p_\mu b(r,\mathbf{p}) + d^\dagger(r,\mathbf{p})p_\mu d(r,\mathbf{p})\right], \tag{7.108}$$

$$\mathbf{J} = \int d\Omega_{\mathbf{p}} \sum_r \left[b^\dagger(r,\mathbf{p})\left(\mathbf{s} - i\mathbf{p}\wedge\frac{\partial}{\partial\mathbf{p}}\right)_{rr'} b(r,\mathbf{p}) + d^\dagger(r,\mathbf{p})\left(\mathbf{s} - i\mathbf{p}\wedge\frac{\partial}{\partial\mathbf{p}}\right)_{rr'} d(r,\mathbf{p})\right], \tag{7.109}$$

$$\mathbf{K} = \int d\Omega_{\mathbf{p}} \sum_r \left[b^\dagger(r,\mathbf{p})\left(\frac{\mathbf{p}\wedge\mathbf{s}}{p^0+m} + ip^0\frac{\partial}{\partial\mathbf{p}}\right)_{rr'} b(r,\mathbf{p}) + d^\dagger(r,\mathbf{p})\left(\frac{\mathbf{p}\wedge\mathbf{s}}{p^0+m} + ip^0\frac{\partial}{\partial\mathbf{p}}\right)_{rr'} d(r,\mathbf{p})\right], \tag{7.110}$$

as can be inferred by Eqs. (7.59)-(7.61).

On the field operators Eqs. (7.66) and (7.67) now give

$$[p_\mu, \varphi(x)] = -i\partial_\mu\varphi(x), \tag{7.111}$$

$$[J_{(\mu\nu)}, \varphi(x)] = -\left[\sigma_{\mu\nu} - i(x_\mu\partial_\nu - x_\nu\partial_\mu)\right]\varphi(x), \tag{7.112}$$

From the point of view of the Poincaré group it is evident from the construction and from the generators (7.108)-(7.110) that the antiparticle states are identical to the particle ones: they describe a system of free particles of mass m.

Appendix A: Commutators

The commutator of two operators A and B is defined as

$$[A, B] = AB - BA. \tag{A1}$$

The commutator satisfies to the following Lie algebra relations

$$[A, A] = 0, \tag{A2}$$

$$[A, B] = -[B, A], \tag{A3}$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \tag{A4}$$

where the third one is known as the Jacobi identity.

For three operators A, B, and C we also have

$$[A, B + C] = [A, B] + [A, C], \tag{A5}$$

$$[A, BC] = B[A, C] + [A, B]C. \tag{A6}$$

If $[A, B] = \alpha \in \mathbb{C}$ then

$$\begin{aligned} [A, B^2] &= B[A, B] + [A, B]B = 2\alpha B, && \text{(A7)} \\ [A, B^3] &= B[A, B^2] + [A, B]B^2 = 3\alpha B^2, && \text{(A8)} \\ &\dots \\ [A, B^n] &= n\alpha B^{n-1}. && \text{(A9)} \end{aligned}$$

Then, given a smooth function f, using its Taylor series expansion, we readily obtain

$$[A, f(B)] = \alpha \frac{df(B)}{dB}. \tag{A10}$$

In general we can prove the following lemma:
Hadamard lemma: Given any two operators A and B we have

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots \tag{A11}$$

Proof: Consider the function $f(s) = e^{sA} B e^{-sA}$. We want $f(1)$. Taylor expand $f(s)$ around $s = 0$

$$f(s) = f(0) + s f'(0) + \frac{1}{2!} s^2 f''(0) + \frac{1}{3!} s^3 f'''(0) + \dots, \tag{A12}$$

but it is easy to see that

$$\begin{aligned} f'(s) &= e^{sA} A B e^{-sA} - e^{sA} B A e^{-sA} = e^{sA}[A, B] e^{-sA}, && \text{(A13)} \\ f''(s) &= e^{sA}[A, [A, B]] e^{-sA}, && \text{(A14)} \\ f'''(s) &= e^{sA}[A, [A, [A, B]]] e^{-sA}, && \text{(A15)} \end{aligned}$$

and so on.

The following theorem is also of great importance:
Theorem: Given two hermitian operators A and B which commutes, $[A, B] = 0$, they can be diagonalized simultaneously on the same orthonormal base of vectors of the Hilbert space.

Appendix B: The Levi-Civita symbol

The Levi-Civita symbol $\epsilon_{i_1 i_2 \dots i_n}$ is defined as a total antisymmetric n rank tensor with $\epsilon_{012\dots n} = 1$.

In two dimensions

$$\begin{aligned} \epsilon_{ij}\epsilon_{ik} &= \delta_{jk}, && \text{(B1)} \\ \epsilon_{ij}\epsilon_{ij} &= 2, && \text{(B2)} \end{aligned}$$

where in the first equation we contracted one index and in the second equation we contracted both indexes.

In three dimensions

$$\begin{aligned} \epsilon_{ijk}\epsilon_{ilm} &= \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, && \text{(B3)} \\ \epsilon_{ijk}\epsilon_{ijl} &= 3\delta_{kl} - \delta_{kl} = 2\delta_{kl}, && \text{(B4)} \\ \epsilon_{ijk}\epsilon_{ijk} &= 6. && \text{(B5)} \end{aligned}$$

In general

$$\epsilon_{i_1 i_2 \dots i_n}\epsilon_{j_1 j_2 \dots j_n} = \det \begin{pmatrix} \delta_{i_1 j_1} & \cdots & \delta_{i_1 j_n} \\ \vdots & \ddots & \\ \delta_{i_n j_1} & & \delta_{i_n j_n} \end{pmatrix}. \tag{B6}$$

Also for an $n \times n$ matrix $\mathbf{A}$ with $(\mathbf{A})_{ij} = a_{ij}$ we have

$$\begin{aligned} \det(\mathbf{A}) &= \epsilon_{i_1 i_2 \dots i_n} a_{1 i_1} a_{2 i_2} \cdots a_{n i_n}, && \text{(B7)} \\ \det(\mathbf{A})\epsilon_{j_1 j_2 \dots j_n} &= \epsilon_{i_1 i_2 \dots i_n} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}. && \text{(B8)} \end{aligned}$$

Appendix C: Angular momentum

Consider the angular momentum hermitian operator $\widehat{\mathbf{L}}$, where the hat denotes the operator. Then the following commutation relations hold

$$[\widehat{L}_i, \widehat{L}_j] \; = \; i\epsilon_{ijk}\widehat{L}_k. \tag{C1}$$

Then define

$$\widehat{L^2} \; = \; \sum_{i=1}^{3} \widehat{L}_i^2, \tag{C2}$$

$$\widehat{L}_\pm \; = \; \widehat{L}_1 \pm \widehat{L}_2. \tag{C3}$$

We can then prove the following relations

$$[\widehat{L^2}, \widehat{L}_i] \; = \; 0, \tag{C4}$$

$$[\widehat{L}_+, \widehat{L}_-] \; = \; 2\widehat{L}_3, \tag{C5}$$

$$[\widehat{L}_3, \widehat{L}_\pm] \; = \; \pm\widehat{L}_\pm, \tag{C6}$$

and

$$\widehat{L^2} = \widehat{L}_+\widehat{L}_- + \widehat{L}_3^2 - \widehat{L}_3 = \widehat{L}_-\widehat{L}_+ + \widehat{L}_3^2 + \widehat{L}_3 \tag{C7}$$

Since $\widehat{L^2}$ commutes with $\widehat{L}_3$ we can diagonalize them simultaneously so that

$$\widehat{L^2}|\psi_{L,M}\rangle \; = \; \mathcal{L}^2|\psi_{L,M}\rangle, \tag{C8}$$

$$\widehat{L}_3|\psi_{L,M}\rangle \; = \; M|\psi_{L,M}\rangle, \tag{C9}$$

where, since $\widehat{L^2} - \widehat{L}_3^2 = \widehat{L}_1^2 + \widehat{L}_2^2$, we called L the maximum value of $|M|$ for a given value $\mathcal{L}$. Then

$$\widehat{L}_3\widehat{L}_\pm|\psi_{L,M}\rangle \; = \; (M \pm 1)\widehat{L}_\pm|\psi_{L,M}\rangle, \tag{C10}$$

$$\widehat{L}_+\psi_{L,M} \; = \; 0. \tag{C11}$$

From Eq. (C7) follows

$$0 = \widehat{L}_-\widehat{L}_+|\psi_{L,M}\rangle = (\widehat{L^2} - \widehat{L}_3^2 - \widehat{L}_3)|\psi_{L,M}\rangle, \tag{C12}$$

or $\mathcal{L}^2 = L(L+1)$. Also M can assume $2L+1$ values, namely $M = L, L-1, \ldots, -L$. And $2L = 0, 1, 2, 3, \ldots$.

For the orbital angular momentum $\widehat{\mathbf{L}} = \widehat{\mathbf{r}} \wedge \widehat{\mathbf{p}}$. In the coordinate representation $\widehat{\mathbf{r}} = \mathbf{r}$ and $\widehat{\mathbf{p}} = -i\boldsymbol{\nabla}_\mathbf{r}$. From the commutation relations for position and momentum

$$[\widehat{r}_i, \widehat{r}_j] \; = \; 0, \tag{C13}$$

$$[\widehat{p}_i, \widehat{p}_j] \; = \; 0, \tag{C14}$$

$$[\widehat{r}_i, \widehat{p}_j] \; = \; i\delta_{ij}, \tag{C15}$$

follows

$$[\widehat{L}_i, \widehat{r}_j] \; = \; i\epsilon_{ijk}\widehat{r}_k, \tag{C16}$$

$$[\widehat{L}_i, \widehat{p}_j] \; = \; i\epsilon_{ijk}\widehat{p}_k, \tag{C17}$$

and again Eq. (C1). Using spherical coordinates

$$r_1 = r\sin\theta\cos\phi, \quad r_2 = r\sin\theta\sin\phi, \quad r_3 = r\cos\theta, \tag{C18}$$

we find in particular

$$\widehat{L}_3 = -i\frac{\partial}{\partial\phi}. \tag{C19}$$

So we see that the eigenvalue equation

$$\widehat{L}_3\psi_{L,M}(\mathbf{r}) = M\psi_{L,M}(\mathbf{r}), \tag{C20}$$

has solution

$$\psi_{L,M} = f(r,\theta)e^{iM\phi}, \tag{C21}$$

where f is an arbitrary function. If the function $\psi_{L,M}$ has to be single valued, it must be periodic in ϕ with period 2π. Hence we find that additionally for the orbital case we must have $M = 0, \pm 1, \pm 2, \ldots$, i.e. L must be an integer.

If we have to add the angular momentum of two different systems, $\widehat{L} = \widehat{L^{(1)}} + \widehat{L^{(2)}}$, we can either choose the set of commuting operators $\{\widehat{(L^{(1)})^2}, \widehat{(L^{(2)})^2}, \widehat{L^{(1)}}_3, \widehat{L^{(2)}}_3\}$ or the other one $\{\widehat{(L^{(1)})^2}, \widehat{(L^{(2)})^2}, \widehat{L}^2, \widehat{L}_3\}$, since $[\widehat{L^{(1)}}, \widehat{L^{(2)}}] = 0$.

Appendix D: SU(2)

The special unitary group of degree n, SU(n), is the group of $n \times n$ unitary matrices with determinant 1. Its dimension as a real manifold is $n^2 - 1 = 3$. Topologically it is compact and simply connected. Algebraically it is a simple Lie group.

Consider the 2×2 complex matrices A which are unitary $A^\dagger A = 1$ and with determinant equal to 1. The most general 2×2 complex matrix can be written as

$$A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \quad z_i = \rho_i e^{i\varphi_i}. \tag{D1}$$

Imposing unitarity is the same as imposing the three following conditions

$$z_1^* z_1 + z_3^* z_3 = 1, \tag{D2}$$
$$z_2^* z_2 + z_4^* z_4 = 1, \tag{D3}$$
$$z_1^* z_2 + z_3^* z_4 = 0. \tag{D4}$$

Imposing that the determinant is 1 amounts to setting

$$z_1 z_4 - z_2 z_3 = 1. \tag{D5}$$

This four conditions can be rewritten as follows

$$\rho_1^2 + \rho_3^2 = 1, \tag{D6}$$
$$\rho_2^2 + \rho_4^2 = 1, \tag{D7}$$
$$\rho_1\rho_2 e^{i(\varphi_2-\varphi_1)} + \rho_3\rho_4 e^{i(\varphi_4-\varphi_3)} = 0, \tag{D8}$$
$$\rho_1\rho_4 e^{i(\varphi_1+\varphi_4)} - \rho_2\rho_3 e^{i(\varphi_2-\varphi_3)} = 1. \tag{D9}$$

Taking the modulus of Eq. (D8) gives $\rho_1\rho_2 = \rho_3\rho_4$. When we use this relation in Eqs. (D6) and (D7) we find $\rho_1 = \rho_4$ and $\rho_2 = \rho_3$. Then Eq. (D8) gives $\varphi_2 - \varphi_1 + \varphi_3 - \varphi_4 = \pi$ which when used in Eq. (D9) gives

$$\rho_1^2 + \rho_2^2 = e^{-i(\varphi_1+\varphi_4)}, \tag{D10}$$

which in turn is satisfied by $\rho_1^2 + \rho_2^2 = 1$ and $\varphi_1 + \varphi_4 = 0$. Then we end up with matrices of the form

$$A = \begin{pmatrix} \rho_1 e^{i\varphi_1} & \pm\sqrt{1-\rho_1}e^{i\varphi_2} \\ \mp\sqrt{1-\rho_1}e^{-i\varphi_2} & \rho_1 e^{-i\varphi_1} \end{pmatrix}. \tag{D11}$$

In other words we can say that

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1 \right\}. \tag{D12}$$

The Lie algebra $\mathcal{SU}(2)$ of the group is obtained through the exponential map as the 2×2 complex matrices ia such that $A = e^{ia}$. Then the unitarity of A implies that a be hermitian and the condition for A to have determinant 1 implies that a be traceless. It is easy to prove that $\mathcal{SU}(n)$ has dimension $2n(n-1)/2 + n - 1 = n^2 - 1$ and

$$\mathcal{SU}(2) = \{i\boldsymbol{\theta}\cdot\boldsymbol{\sigma} \mid \boldsymbol{\theta} \in \mathbb{R}^3\}, \tag{D13}$$

with σ_i the Pauli matrices

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{D14}$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{D15}$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{D16}$$

If we add to the Pauli matrices the identity matrix

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3, \tag{D17}$$

we obtain a base for the vector space of hermitian 2×2 complex matrices.

The Pauli matrices are unitary and some of their properties are as follows

$$\det(\sigma_i) = -1, \tag{D18}$$

$$\mathrm{Tr}(\sigma_i) = 0, \tag{D19}$$

$$\det(\mathbf{a}\cdot\boldsymbol{\sigma}) = -|\mathbf{a}|^2, \tag{D20}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \tag{D21}$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}1, \tag{D22}$$

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = (\mathbf{a}\cdot\mathbf{b})1 + i(\mathbf{a}\wedge\mathbf{b})\cdot\boldsymbol{\sigma}, \tag{D23}$$

$$e^{ia(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})} = 1\cos a + i(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})\sin a. \tag{D24}$$

The Pauli matrices offer a representation for the spin 1/2 operator $\mathbf{s}$ as follows

$$\mathbf{s} = \frac{\boldsymbol{\sigma}}{2}. \tag{D25}$$

There exists a $2:1$ group homomorphism between $SU(2)$ and $SO(3)$.

Appendix E: Velocity transformations

A velocity transformation with $\boldsymbol{\beta} = (0, 0, \beta)$ is $x' = \Lambda x$ with

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{E1}$$

where $\gamma = 1/\sqrt{1-\beta^2}$. The velocity transformation can be cast into another useful form by defining a parameter α called the *rapidity* (or hyperbolic angle) such that

$$e^{\alpha} = \gamma(1+\beta) = \sqrt{\frac{1+\beta}{1-\beta}}, \tag{E2}$$

and thus

$$e^{-\alpha} = \gamma(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}}. \tag{E3}$$

So

$$\gamma \;=\; \cosh\alpha = \frac{e^{\alpha}+e^{-\alpha}}{2}, \tag{E4}$$

$$\beta\gamma \;=\; \sinh\alpha = \frac{e^{\alpha}-e^{-\alpha}}{2}, \tag{E5}$$

$$\tag{E6}$$

and therefore

$$\beta = \tanh\alpha. \tag{E7}$$

We then have

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\alpha & 0 & 0 & -\sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{E8}$$

with

$$\begin{pmatrix} \cosh\alpha & 0 & 0 & -\sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} = \exp\left[-i\alpha \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\right] \equiv \exp(-i\alpha K^3), \tag{E9}$$

where the simpler Lie-algebraic hyperbolic rotation generator iK^3 is called a *boost* generator.

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