Lectures on the renormalization group in statistical physics

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We review some of the ideas of the renormalization group in the statistical physics of classical and quantum fluids theory. The origin, the nature, the basis, the formulation, the critical exponents and scaling, relevance, irrelevance, and marginality, universality, and Wilson's concept of flows and fixed point in a space of Hamiltonians.

CONTENTS

In a recent Review of Modern Physics, M. E. Fisher (Fisher, 1998) presented, to a wide audience, the ideas of the Renormalization Group (RG) theory behind statistical mechanics of matter physics and Quantum Field Theory (QFT).

We will also follow the lectures given by N. Goldenfeld (Goldenfeld, 1992) at the University of Illinois at Urbana-Champaign in 1992.

Despite its name the theory is not really about a group but about a *semigroup* since the set of transformations involved is not necessarily invertible. The theory is thought as one of the underlying ideas in the theoretical structure of QFT even if the roots of RG theory has to be looked upon the theory of critical phenomena of the statistical mechanics of matter physics.

I. NOTATION

In spIn specifying critical behavior (and asymptotic variation more generally) a little more precision than normally used is really called for. Following well established custom, we use $\simeq$ for "approximately equals" in a rough and ready

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sense, as in $\pi^2 \simeq 10$. But to express "$f(x)$ varies like x^λ when x is small and positive" i.e., just to specify a critical exponent, we write:

$$f(x) \sim x^\lambda \quad (x \to 0^+). \tag{1.1}$$

Then the precise implication is

$$\lim_{x\to 0^+} \ln|f(x)|/\ln x = \lambda. \tag{1.2}$$

We define $\approx$ as "asymptotically equals" so that

$$f(x) \approx g(x) \quad (x \to 0^+), \tag{1.3}$$

implies

$$\lim_{x\to 0^+} f(x)/g(x) = 1. \tag{1.4}$$

We define the $o(\cdot)$ symbol as follows:

$$f = o(g) \quad (x \to 0), \tag{1.5}$$

means that $|f| < c|g|$ for some constant c and $|x|$ small enough.

II. THE ORIGIN OF RG

The history of the RG has to be reckoned on the work of Lev D. Landau who can be regarded as the founder of systematic *effective field theories* and of the concept of the *order parameter* ((Landau and Lifshitz, 1958) sec. 135). That is one recognizes that there is a microscopic level of description and believes it should have certain general, overall properties especially as regards locality and symmetry. Those then serve to govern the most characteristic behavior on scales greater than atomic. Known the nature of the order parameter, suppose, for example, it is a complex number and like a wave function, then one knows much about the macroscopic nature of a physical system.

Traditionally, one characterizes statistical mechanics as directly linking the *microscopic* world of nuclei and atoms (on length scales of 10^{-13} to 10^{-8} cm) to the *macroscopic* world of say, millimeters to meters. But the order parameter, as a dynamic, fluctuating object in many cases intervenes on an intermediate or *mesoscopic* level characterized by scales of tens or hundreds of angstroms up to microns. A major collaborator of Landau and developer of the concept was V. L. Ginzburg (Ginzburg, 1997; Ginzburg and Landau, 1959) in particular for the theory of superconductivity.

Landau's concept of the order parameter brought light, clarity, and form to the general theory of phase transitions, leading eventually, to the characterization of multicritical points and the understanding of many characteristic features of ordered states. But in 1944 Lars Onsager, by a mathematical tour de force, computed exactly the partition function and thermodynamic properties of the simplest model of a ferromagnet or a fluid (Kaufman and Onsager, 1949; Onsager, 1944, 1949). This model, the Ising model, exhibited a sharp critical point: But the explicit properties, in particular, the nature of the critical singularities disagreed profoundly with essentially all the detailed predictions of the Landau theory (and of all foregoing, more specific theories). From this challenge, and from experimental evidence pointing in the same direction (Fisher, 1965), grew the ideas of universal but nontrivial critical exponents (Domb, 1960, 1996), special relations between different exponents (Essam and Fisher, 1963), and then, scaling descriptions of the region of a critical point (Domb and Hunter, 1965; Kadanoff, 1966; Patashinskii and Pokrovskii, 1966; Widom, 1965a,b). These insights served as stimulus and inspiration to Kenneth Wilson in his pursuit of an understanding of QFTs (Wilson, 1983). Indeed, once one understood the close mathematical analogy between doing statistical mechanics with effective Hamiltonians and doing quantum field theory (especially with the aid of Feynman's path integral) the connections seemed almost obvious. Needless to say, however, the realization of the analogy did not come overnight: In fact, Wilson himself was the individual who first understood clearly the analogies at the deepest levels.

In 1971, Wilson, having struggled with the problem of the systematic integrating out of appropriate degrees of freedom and the resulting RG flows for four or five years, was able to cast his RG ideas into a conceptually effective framework (Wilson, 1971a,b, 1983). Effective in the sense that one could do certain calculations with it. And Franz Wegner, very soon afterwards (Wegner, 1972a,b), further clarified the foundations and exposed their depth and breadth. An early paper by Kadanoff and Wegner (Kadanoff and Wegner, 1971) showing when and how universality could fail was particularly significant in demonstrating the richness of Wilson's conception. Their focus on *relevant*, *irrelevant*, and *marginal* operators (or perturbations) has played a central role (Kadanoff, 1976; Wegner, 1976). The advent of Wilson's concept of the RG gave more precise meaning to the effective ("coarse-grained") Hamiltonians that stemmed from the work of Landau and Ginzburg. One now pictures the Landau-Ginzburg-Wilson (LGW) Hamiltonians as true but significantly renormalized Hamiltonians in which finer microscopic degrees of freedom have been integrated out.

So our understanding of "anomalous" i.e., non-Landau-type but, in reality, standard critical behaviour was greatly enhanced. The epsilon expansion (see chapter 12 of the Goldenfeld book (Goldenfeld, 1992)), which used as a small, perturbation parameter the deviation of the spatial dimensionality, d, from four dimensions, namely, $\epsilon = 4 - d$, provided a powerful and timely tool (Wilson and Fisher, 1972). It had the added advantage, if one wanted to move ahead, that the method looked something like a cookbook so that "any fool" could do or check the calculations, whether they really understood, at a deeper level, what they were doing or not. But in practice that also has a real benefit in that a lot of calculations do get done, and some of them turn up new and interesting things or answer old or new questions in instructive ways. A few calculations reveal apparent paradoxes and problems which serve to teach one and advance understanding.

The foundations of RG theory are in the *critical exponent relations* and the crucial *scaling concepts* developed in 1963-66 (Essam and Fisher, 1963; Fisher, 1967a; Kadanoff, 1966; Widom, 1965a,b).

Some antedating reviews on RG theory are to be found in the following Refs. (Domb, 1960; Fisher, 1965, 1967b; Kadanoff *et al.*, 1967; Stanley, 1971). Retrospective reviews can be found in the following books (Baker Jr., 1990; Creswick *et al.*, 1992; Domb, 1996). Introductory accounts in an informal lecture style are presented by M. E. Fisher in Refs. (Fisher, 1965, 1983).

III. THE DECAY OF CORRELATION FUNCTIONS

Consider a locally defined microscopic variable which we will denote $\psi(\mathbf{r})$. In a ferromagnet this might well be the local magnetization, $M(\mathbf{r})$, or spin vector, $S(\mathbf{r})$, at point $\mathbf{r}$ in ordinary d-dimensional (Euclidean) space; in a fluid it might be the deviation $\delta\rho(\mathbf{r})$, of the fluctuating density at $\mathbf{r}$ from the mean density. In QFT the local variables $\psi(r)$ are the basic quantum fields which are "operator valued". For a magnetic system, in which quantum mechanics was important, $M(\mathbf{r})$ and $S(\mathbf{r})$ would, likewise, be operators. However, the distinction is of relatively minor importance so that we may, for ease, suppose $\psi(\mathbf{r})$ is a simple classical variable. It will be most interesting when ψ is closely related to the order parameter for the phase transition and critical behavior of concern.

By means of a scattering experiment (using light, x rays, neutrons, electrons, etc.) one can often observe the corresponding pair correlation function (or basic "two-point function")

$$G(\mathbf{r}) = \langle \psi(\mathbf{0})\psi(\mathbf{r}) \rangle, \tag{3.1}$$

where the angular brackets $\langle \cdot \rangle$ denote a statistical average over the thermal fluctuations that characterize all equilibrium systems at nonzero temperature. (Also understood, when $\psi(\mathbf{r})$ is an operator, are the corresponding quantum-mechanical expectation values).

Physically, $G(\mathbf{r})$ is important since it provides a direct measure of the influence of the leading microscopic fluctuations at the origin $\mathbf{0}$ on the behavior at a point distance $r = |\mathbf{r}|$ away. But, almost by definition, in the vicinity of an appropriate critical point, for example the Curie point of a ferromagnet when $\psi = M$ or the gas-liquid critical point when $\psi = \delta\rho$, a strong "ordering" influence or correlation spreads out over, essentially, macroscopic distances. As a consequence, precisely at criticality one rather generally finds a power-law decay, namely,

$$G_c(\mathbf{r}) \approx D/r^{d-2+\eta} \quad \text{as} \quad r \to \infty, \tag{3.2}$$

which is characterized by the critical exponent (or critical index) $d - 2 + \eta$.

Now all the theories one first encounters, the so-called "classical" or Landau-Ginzburg or van der Waals theories, etc., predict, quite unequivocally, that η vanishes. In QFT this corresponds to the behavior of a free massless particle. Mathematically, the reason underlying this prediction is that the basic functions entering the theory have (or are assumed to have) a smooth, analytic, nonsingular character so that, following Newton, they may be freely differentiated and, thereby expanded in Taylor series with positive integral powers even at the critical point. In QFT the classical exponent value $d - 2$ (implying $\eta = 0$) can often be determined by naive dimensional analysis or "power counting": Then $d - 2$ is said to represent the "canonical dimension" while η, if nonvanishing, represents the "dimensional anomaly". Physically, the prediction $\eta = 0$ typically results from a neglect of fluctuations or, more precisely as Wilson emphasized, from the assumption that only fluctuations on much smaller scales can play a significant role: In such circumstances the fluctuations can be safely incorporated into effective (or renormalized) parameters (masses, coupling constants, etc.) with no change in the basic character of the theory.

But a power-law dependence on distance implies a *lack* of a definite length scale and, hence, a *scale invariance*. To illustrate this, let us rescale distances by a factor b so that $\mathbf{r} \to \mathbf{r}' = b\mathbf{r}$, and, at the same time, rescale the order parameter ψ by some "covariant" factor b^ω where ω will be a critical exponent characterizing ψ. Then we have that if one has $\omega = \frac{1}{2}(d - 2 + \eta)$, the factors of b drop out and the form in Eq. (3.2) is recaptured. In other words $G_c(\mathbf{r})$ is *scale invariant* (or covariant): Its variation reveals no characteristic lengths, large, small, or intermediate.

Since power laws imply scale invariance and the absence of well separated scales, the classical theories should be suspect at (and near) criticality. Indeed, one finds that the "anomaly" h does *not* normally vanish (at least for dimensions d less than 4, which is the only concern in a physics of matter laboratory). In particular, from the work of Kaufman and Onsager (Kaufman and Onsager, 1949) one can show analytically that $\eta = \frac{1}{4}$ for the $d = 2$ Ising model. Consequently, the analyticity and Taylor expansions presupposed in the classical theories are *not* valid. Therein lies the challenge to theory. Indeed, it proved hard even to envisage the nature of a theory that would lead to $\eta \neq 0$. The power of the renormalization group is that it provides a conceptual and, in many cases, a computational framework within which anomalous values for η (and for other exponents like ω and its analogs for all local quantities such as the energy density $\mathcal{E}$) arise naturally.

In applications to matter physics, it is clear that the power law in Eq. (3.2) can hold only for distances relatively large compared to atomic lengths or lattice spacings which we will denote a. In this sense the scale invariance of correlation functions is only *asymptotic* hence the symbol $\approx$, for "asymptotically equals", and the proviso $r \to \infty$ in Eq. (3.2). A more detailed description would account for the effects of nonvanishing a, at least in leading order. By contrast, in QFT the microscopic distance a represents an "ultraviolet" cutoff which, since it is in general unknown, one normally wishes to remove from the theory. If this removal is not done with surgical care, which is what the renormalization program in QFT is all about, the theory remains plagued with infinite divergencies arising when $a \to 0$, i.e., when the "cutoff is removed". But in statistical physics one always anticipates a short-distance cutoff that sets certain physical parameters such as the value of T_c; infinite terms *per se* do not arise and certainly do not drive the theory as in QFT.

One may, however, provide a more concrete illustration of scale dependence by referring again to the power law Eq. (3.2). If the exponent η vanishes, or equivalently, if ψ has its canonical dimension, so that $\omega = \omega_{\text{can}} = \frac{1}{2}(d-2)$, one may regard the amplitude D as a fixed, measurable parameter which will typically embody some real physical significance. Suppose, however, η does not vanish but is nonetheless relatively small: Indeed, for many $(d = 3)$-dimensional systems, one has $\eta \simeq 0.035$ (Baker Jr., 1990; Domb, 1996; Fisher, 1983; Fisher and Burford, 1967). Then we can introduce a "renormalized" or "scale-dependent" parameter

$$\widetilde{D}(r) \approx D/r^\eta \quad \text{as} \quad r \to \infty, \tag{3.3}$$

and rewrite the original result simply as

$$G_c(r) = \widetilde{D}(r)/r^{d-2}. \tag{3.4}$$

Since η is small we see that $\widetilde{D}(r)$ varies slowly with the scale r on which it is measured. In many cases in QFT the dimensions of the field ψ (alias the order parameter) are subject only to marginal perturbations (see below) which translate into a $\ln r$ dependence of the renormalized parameter $\widetilde{D}(r)$; the variation with scale is then still weaker than when $\eta \neq 0$.

IV. THE CHALLANGES POSED BY CRITICAL PHENOMENA

Physics is an experimental science. So let us briefly review a few experimental findings that serves to focus attention on the principal theoretical challenges faced by, and rather fully met by RG theory.

In 1869 Andrews reported to the Royal Society his observations of carbon dioxide sealed in a (strong) glass tube at a mean overall density, ρ, close to 0.5 gm cm^{-3}. At room temperatures the fluid breaks into two phases: A liquid of density $\rho_{\text{liq}}(T)$ that coexists with a lighter vapor or gas phase of density $\rho_{\text{gas}}(T)$ from which it is separated by a visible meniscus or interface; but when the temperature, T, is raised and reaches a sharp critical temperature, $T_c \simeq 31.04$ °C, the liquid and gaseous phases become identical, assuming a common density $\rho_{\text{liq}} = \rho_{\text{gas}} = \rho_c$ while the meniscus disappears in a "mist" of "critical opalescence". For all T above T_c there is a complete "continuity of state", i.e., no distinction whatsoever remains between liquid and gas (and there is no meniscus). A plot of $\rho_{\text{liq}}(T)$ and $\rho_{\text{gas}}(T)$, as illustrated somewhat schematically in Fig. 1(d), represents the so-called gas-liquid *coexistence curve* or *binodal*: The

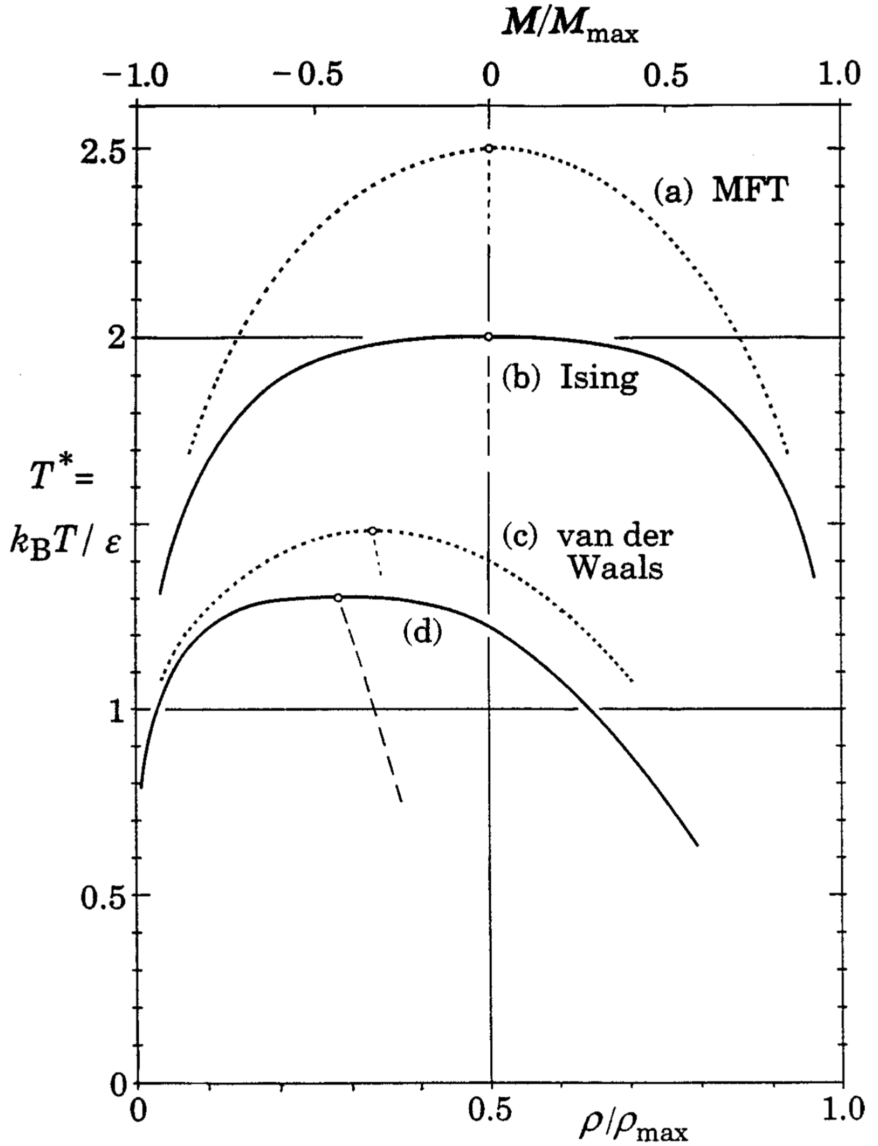

FIG. 1 Temperature variation of gas-liquid coexistence curves (temperature, T, versus density, ρ) and corresponding spontaneous magnetization plots (magnetization, M, versus T). The solid curves, (b) and (d), represent (semiquantitatively) observation and modern theory, while the dotted curves (a) and (c) illustrate the corresponding "classical" predictions (mean-field theory and van der Waals approximation). These latter plots are parabolic through the critical points (small open circles) instead of obeying a power law with the universal exponent $\beta \simeq 0.325$: See Eqs. (4.3) and (11). The energy scale ε, and the maximal density and magnetization, $\rho_{\rm max}$ and $M_{\rm max}$, are nonuniversal parameters particular to each physical system; they vary widely in magnitude.

two halves, $\rho_{\rm liq} > \rho_c$ and $\rho_{\rm gas} < \rho_c$, meet smoothly at the critical point (T_c, ρ_c), shown as a small circle in Fig. 1: The dashed line below T_c represents the diameter defined by $\rho(T) = \frac{1}{2}[\rho_{\rm liq}(T) + \rho_{\rm gas}(T)]$.

The same phenomena occur in all elemental and simple molecular fluids and in fluid mixtures. The values of T_c , however, vary widely: e.g., for helium-four one finds 5.20 K while for mercury $T_c \simeq 1764$ K. The same is true for the critical densities and concentrations: These are thus "nonuniversal parameters" directly reflecting the atomic and molecular properties, i.e., the physics on the scale of the cutoff a. Hence, in Fig. 1, $\rho_{\rm max}$ (which may be taken as the density of the corresponding crystal at low T) is of order $1/a^3$, while the scale of k_BT_c is set by the basic microscopic potential energy of attraction denoted ε. While of considerable chemical, physical, and engineering interest, such parameters will be of marginal concern to us here. The point, rather, is that the shapes of the coexistence curves, $\rho_{\rm liq}(T)$ and $\rho_{\rm gas}(T)$ versus T, become asymptotically universal in character as the critical point is approached.

To be more explicit, note first an issue of symmetry. In QFT, symmetries of many sorts play an important role: They may (or must) be built into the theory but can be "broken" in the physically realized vacuum state(s) of the quantum field. In the physics of fluids the opposite situation pertains. There is no real physical symmetry between coexisting liquid and gas: They are just different states, one a relatively dense collection of atoms or molecules, the other a relatively dilute collection, see Fig. 1(d). However, if one compares the two sides of the coexistence curve, gas and liquid, by forming the ratio

$$R(T) = [\rho_c - \rho_{\rm gas}(T)]/[\rho_c - \rho_{\rm liq}(T)], \tag{4.1}$$

one discovers an extraordinarily precise asymptotic symmetry. Explicitly, when T approaches T_c from below or, introducing a convenient notation,

$$t \equiv (T - T_c)/T_c \to 0^-, \tag{4.2}$$

one finds $R(T) \to 1$. This simply means that the physical fluid builds for itself an exact mirror symmetry in density (and other properties) *as* the critical point is *approached*. And this is a universal feature for all fluids near criticality. (This symmetry is reflected in Fig. 1(d) by the high, although not absolutely perfect, degree of asymptotic linearity of the coexistence-curve diameter, $\overline{\rho}(T)$, the dashed line described above).

More striking than the (asymptotic) symmetry of the coexistence curve is the universality of its shape close to T_c, visible in Fig. 1(d) as a flattening of the graph relative to the parabolic shape of the corresponding classical prediction, see plot (c) in Fig. 1, which is derived from the famous van der Waals equation of state. Rather generally one can describe the shape of a fluid coexistence curve in the critical region via the power law

$$\Delta\rho \equiv \frac{1}{2}[\rho_{\rm liq}(T) - \rho_{\rm gas}(T)] \approx B|t|^\beta \quad \text{as} \quad t \to 0^-, \tag{4.3}$$

where B is a *non*universal amplitude while the critical exponent β takes a *universal* value

$$\beta \simeq 0.325, \tag{4.4}$$

(in which the last figure is uncertain). To stress the point: β is a nontrivial number, not known exactly, but it is the same for all fluid critical points! This contrasts starkly with the classical prediction $\beta = \frac{1}{2}$ [corresponding to a parabola: See Fig. 1(c)]. The value in Eq. (4.4) applies to ($d = 3$)-dimensional systems. Classical theories make the same predictions for all d. On the other hand, for $d = 2$, Onsager's work (Onsager, 1949) on the square-lattice Ising model leads to $\beta = \frac{1}{8}$. This value has since been confirmed experimentally by Kim and Chan (Kim and Chan, 1984) for a "two-dimensional fluid" of methane (CH_4) adsorbed on the flat, hexagonal-lattice surface of graphite crystals.

Not only does the value in Eq. (4.4) for β describe many types of fluid system, it also applies to anisotropic magnetic materials, in particular to those of Ising-type with one "easy axis". For that case, in vanishing magnetic fields, H, below the Curie or critical temperature, T_c , a ferromagnet exhibits a spontaneous magnetization and one has $M = \pm M_0(T)$. The sign, + or −, depends on whether one lets H approach zero from positive or negative values. Since, in equilibrium, there is a full, natural physical symmetry under $H \to -H$ and $M \to -M$ (in contrast to fluid systems) one clearly has $M_c = 0$: Likewise, the asymptotic symmetry corresponding to Eq. (4.1) is, in this case exact for all T: See Fig. 1, plots (a) and (b). Thus, as is evident in Fig. 1, the global shape of a spontaneous magnetization curve does not closely resemble a normal fluid coexistence curve. Nevertheless, in the asymptotic law

$$M_0(T) \approx B|t|^\beta \quad \text{as} \quad t \to 0^-, \tag{4.5}$$

the exponent value in Eq. (4.4) still applies for $d = 3$: See Fig. 1(b); the corresponding classical "mean-field theory" in plot (a), again predicts $\beta = \frac{1}{2}$. For $d = 2$ the value $\beta = \frac{1}{8}$ is once more valid.

And, beyond fluids and anisotropic ferromagnets many other systems belong, more correctly their critical behavior belongs, to the "Ising universality class". Included are other magnetic materials (antiferromagnets and ferrimagnets), binary metallic alloys (exhibiting order-disorder transitions), certain types of ferroelectrics, and so on.

For each of these systems there is an appropriate order parameter and, via Eq. (3.2), one can then define (and usually measure) the correlation decay exponent η which is likewise universal. Indeed, essentially any measurable property of a physical system displays a universal critical singularity. Of particular importance is the exponent $\alpha \simeq 0.11$ (Ising, $d = 3$) which describes the divergence to infinity of the specific heat via

$$c(T) \approx A^\pm/|t|^\alpha \quad \text{as} \quad t \to 0^\pm, \tag{4.6}$$

(at constant volume for fluids or in zero field, $H = 0$, for ferromagnets, etc.). The amplitudes A^+ and A^- are again nonuniversal; but their dimensionless ratio, A^+/A^-, is universal, taking a value close to 0.52. When $d = 2$, as Onsager (Onsager, 1944) found, $A^+/A^- = 1$ and $|t|^{-\alpha}$ is replaced by $\ln|t|$. But classical theory merely predicts a jump in specific heat, $\Delta c = c_c^- - c_c^+ > 0$ for all d.

Two other central quantities are a divergent isothermal compressibility $\chi(T)$ (for a fluid) or isothermal susceptibility, $\chi(T) \propto (\partial M/\partial H)_T$ (for a ferromagnet) and, for all systems, a *divergent correlation length*, $\xi(T)$, which measures the growth of the "range of influence" or of correlation observed say, via the decay of the correlation function $G(\mathbf{r}; T)$, see Eq. (3.1) above, to its long-distance limit. For these functions we write

$$\chi(T) \approx C^{\pm}/|t|^{\gamma} \quad \text{and} \quad \xi(t) \approx \xi_0^{\pm}/|t|^{\nu}, \tag{4.7}$$

as $t \to 0^{\pm}$, and find, for $d = 3$ Ising-type systems,

$$\gamma \simeq 1.24 \quad \text{and} \quad \nu \simeq 0.63, \tag{4.8}$$

(while $\gamma = 1\frac{3}{4}$ and $\nu = 1$ for $d = 2$).

As hinted, there are other universality classes known theoretically although relatively few are found experimentally (Aharony, 1976; Fisher, 1974b). Indeed, one of the early successes of RG theory was delineating and sharpening our grasp of the various important universality classes. To a significant degree one found that only the vectorial or tensorial character of the relevant order parameter (e.g., scalar, complex number alias two-component vector, threecomponent vector, etc.) plays a role in determining the universality class. But the whys and the wherefores of this self-same issue represent, as does the universality itself, a prime challenge to any theory of critical phenomena.

V. THE CRITICAL EXPONENTS

It has been believed for a long time that the critical exponents were the same above and below the critical temperature. It has now been shown that this is not necessarily true: When a continuous symmetry is explicitly broken down to a discrete symmetry by irrelevant (in the renormalization group sense) anisotropies, then the exponents γ and γ' are not identical (Leonard and Delamotte, 2015). Here we indicate with a prime the critical exponents for $t < 0$ (ordered phase) and without the prime the critical exponent for $t > 0$ (disordered phase).

A. The classical exponent values

The classical Landau theory (aka mean-field theory) values of the critical exponents for a scalar field are given by (see chapter 5 of Goldenfeld book (Goldenfeld, 1992))

$$\alpha = 0, \tag{5.1}$$

$$\beta = \frac{1}{2}, \tag{5.2}$$

$$\gamma = 1, \tag{5.3}$$

$$\delta = 3, \tag{5.4}$$

adding derivative terms turning it into a mean-field Ginzburg-Landau theory, we get

$$\eta = 0, \tag{5.5}$$

$$\nu = \frac{1}{2}. \tag{5.6}$$

They are valid for $d > d_{uc} = 4$, the upper critical dimension (Fisher, 1974a,b, 1983; Wilson and Fisher, 1972; Wilson and Kogut, 1974).

The problem with mean-field theory is that the critical exponents do not depend on the space dimension. This leads to a quantitative discrepancy in space dimensions 2 and 3, where the true critical exponents differ from the mean-field values. It leads to a qualitative discrepancy in space dimension 1, where a critical point in fact no longer exists, even though mean-field theory still predicts there is one. The space dimension where mean-field theory becomes qualitatively incorrect is called the lower critical dimension.

B. The Ising exponent values

We list in Table I the critical exponents of the ferromagnetic transition in the Ising model (see also Goldenfeld book (Goldenfeld, 1992) p. 111).

TABLE I This table lists the critical exponents of the ferromagnetic transition in the Ising model. In statistical physics, the Ising model describes a continuous phase transition with scalar order parameter. The critical exponents of the transition are universal values and characterize the singular properties of physical quantities. The ferromagnetic transition of the Ising model establishes an important universality class, which contains a variety of phase transitions as different as ferromagnetism close to the Curie point and critical opalescence of liquid near its critical point.

	$d = 2$	$d = 3$	$d = 4$
α	0	0.11008(1)	0
β	1/8	0.326419(3)	1/2
γ	7/4	1.237075(10)	1
δ	15	4.78984(1)	3
η	1/4	0.036298(2)	0
ν	1	0.629971(4)	1/2
ω	2	0.82966(9)	0

C. Exponent relations

Critical exponents obey the following *exponent relations* independently of the universality class

$$\nu d = 2 - \alpha = 2\beta + \gamma = \beta(\delta + 1) = \gamma \frac{\delta + 1}{\delta - 1}, \tag{5.7}$$

$$2 - \eta = \frac{\gamma}{\nu} = d \frac{\delta - 1}{\delta + 1}. \tag{5.8}$$

These equations imply that there are only two independent exponents, e.g., ν and η. All this follows from the theory of the RG.

The relations (Essam and Fisher, 1963; Fisher, 1959, 1962, 1964, 1967b)

$$\gamma = (2 - \eta)\nu, \tag{5.9}$$

$$\alpha + 2\beta + \gamma = 2, \tag{5.10}$$

hold *exactly* for the $d = 2$ Ising models and are valid when $d = 3$ to within the experimental accuracy or the numerical precision (of the theoretical estimates (Baker Jr., 1990; Domb, 1996; Fisher, 1967b)). They are even obeyed exactly by the classical exponent values (which, today, we understand as valid for $d > 4$).

The first relation (5.9) pertains just to the basic correlation function $G(\mathbf{r}; T)$ as defined previously in Eq. (3.1). It follows from the assumption (Fisher, 1959, 1962), supported in turn by an examination of the structure of Onsager's matrix solution to the Ising model (Kaufman and Onsager, 1949; Onsager, 1944) that in the critical region all lengths (much larger than the lattice spacing a) scale like the correlation length $\xi(T)$, introduced in Eq. (4.7). Formally one expresses this principle by writing, for $t \to 0$ and $r \to \infty$,

$$G(\mathbf{r}; T) \approx \frac{D}{r^{d-2+\eta}} \mathcal{G}\left(\frac{r}{\xi(T)}\right), \tag{5.11}$$

where, for consistency with (3.2), the scaling function, $\mathcal{G}(x)$, satisfies the normalization condition $\mathcal{G}(0) = 1$. Integrating $\mathbf{r}$ over all space yields the compressibility/susceptibility $\chi(T)$ and, thence, the relation $\gamma = (2 - \eta)\nu$. This scaling law highlights the importance of the correlation length ξ in the critical region, a feature later stressed and developed further, especially by Widom (Widom, 1965a,b), Kadanoff (Kadanoff, 1966, 1976), and Wilson (Wilson, 1983; Wilson and Kogut, 1974). It isworth remarking that in QFT the inverse correlationlength ξ^{-1}, is basically equivalent to the *renormalized mass* of the field ψ: *Masslessness* then equates with *criticality* since $\xi^{-1} \to 0$.

The second relation (5.10) is proven in section VIII.

VI. THE GAUSSIAN MODEL AND THE UPPER CRITICAL DIMENSION

See chapters 6 and 7 of the book of Goldenfeld (Goldenfeld, 1992).

VII. THE TASK OF RG

One would wish the RG theory to:

(i) explain the ubiquity of *power laws* at and near critical points (as opposed to the exponential laws which governs, for example, the decay of correlation in Coulomb liquids (Das *et al.*, 2011; Martin, 1988));

(ii) explain the values of the leading *thermodynamic and correlation exponents*, $\alpha, \beta, \gamma, \delta, \nu, \eta$, and ω;

(iii) clarify why and how the classical values are in error, including the existence of *borderline dimensionalities*, like $d_{uc} = 4$, above which classical theories become valid;

(iv) find the *correction-to-scaling exponent* θ (and, ideally, the higher-order correction exponents);

(v) give a method to compute *crossover exponents*, ϕ, to check for the *relevance* or *irrelevance* of a multitude of possible perturbations;

(vi) give understanding of *universality* with nontrivial exponents;

(vii) give a derivation of *scaling*;

(viii) allow to understand the *breakdown* of universality and scaling in certain circumstances;

(ix) handle effectively *logarithmic* and more exotic dependences on temperature.

We may start by supposing that one has a set of microscopic, fluctuating, mechanical variables: In QFT these would be the various quantum fields, $\psi(\mathbf{r})$, defined at all points in a Euclidean (or Minkowski) space. In statistical physics the phase space variables $PS = \{R^N, P^N\}$ of N particles of coordinates $R^N = \{\mathbf{r}_1, \ldots, \mathbf{r}_N\}$ and momenta $P^N = \{\mathbf{p}_1, \ldots, \mathbf{p}_N\}$ in a volume V.

In terms of the basic variables PS one can form various "local operators" (or "physical quantities" or "observables") like, for a real fluid, the pressure P, the energy density $\mathcal{E}$, the specific heat c, the isothermal compressibility χ, etc. or, for the Ising model, the pressure P, the spontaneous magnetization M, the energy density $\mathcal{E}$, the specific heat c, the isothermal magnetic susceptibility χ, etc. For a mapping between the Ising model and a real fluid see Goldenfeld book (Goldenfeld, 1992) section 2.12.

A physical system of interest is then specified by its *Hamiltonian* $\mathcal{H}[PS; L]$ which is usually just a spatially uniform sum of local operators made up from the phase space operators and the coupling constant $L = \{\mathbf{L}\}$. The crucial function is the "reduced Hamiltonian"

$$\overline{\mathcal{H}}[PS; K] = -\mathcal{H}[PS; L]/k_B T, \tag{7.1}$$

where k_B is Boltzmann constant, T the absolute temperature, and $K = \{T, L\}$, are the various "thermodynamic fields" (or coupling constants in QFT). We may suppose that one or more of the thermodynamic fields, in particular the temperature, can be controlled directly by the experimenter; but others may be "given" since they will, for example, embody details of the physical system that are "fixed by nature".

An important feature of Wilson's approach, however, is to regard any "physical Hamiltonian" as merely specifying a subspace in a very large space of possible (reduced) Hamiltonians, $\mathbb{H}$. This change in perspective proves crucial to the proper formulation of a renormalization group: In principle, it enters also in QFT although in practice, it is usually given little attention.

The partition function will be

$$Z_N[\overline{\mathcal{H}}] = \mathrm{Tr}_N \left\{ e^{\overline{\mathcal{H}}[PS]} \right\}, \tag{7.2}$$

where the trace operator $\mathrm{Tr}_N\{\cdot\}$, denotes a summation or integration over the possible values of all the $2dN$ variables PS. Then the thermodynamics follow from the total free energy density, which is given by

$$f[\overline{\mathcal{H}}] \equiv f(K) = \lim_{N,V\to\infty} \frac{\ln Z_N[\overline{\mathcal{H}}]}{V}, \tag{7.3}$$

where N and V becomes infinite maintaining the ratio $V/N = a^d$ fixed: In QFT this corresponds to an infinite system with an ultraviolet lattice cutoff.

To the degree that one can actually perform the trace operation in Eq. (7.2) for a particular model system and take the "thermodynamic limit" in Eq. (7.3) one will obtain the precise critical exponents, scaling functions, and so on. This was Onsager's (1944) (Onsager, 1944) route in solving the $d = 2$, spin 1/2 Ising models in zero magnetic field. At first sight one then has no need of RG theory. While one knows for sure that $\alpha = 0$ (ln), $\beta = \frac{1}{8}, \gamma = 1\frac{3}{4}, \nu = 1, \eta = \frac{1}{4}, \ldots$ for the planar Ising models one does not know why the exponents have these values. Indeed, the seemingly inevitable mathematical complexities of solving even such physically oversimplified models exactly (Baxter, 1982) serve to conceal almost all traces of general, underlying mechanisms and principles that might "explain" the results. Also, should one ever achieve truly high precision in simulating critical systems on a computer (a prospect which still seems some decades away (Ceperley, 1995)) the same problem would remain. Thus it comes to pass that even a rather crude and approximate solution of a two-dimensional Ising model by a RG method can be truly instructive.

VIII. THE BASIS AND FORMULATION

At the heart of (real space [1]) RG theory there is the renormalization of the spatial scale via $\mathbf{r} \to \mathbf{r}' = b\mathbf{r}$ which produces on the reduced Hamiltonian the following *renormalization transformation*

$$\overline{\mathcal{H}}'[PS'; K'] = \boldsymbol{\mathcal{R}}_b \overline{\mathcal{H}}[PS, K], \tag{8.1}$$

where we have elected to keep track of the spatial rescaling factor, b, as a subscript of the RG operator $\boldsymbol{\mathcal{R}}$. Thus successive renormalizations with scaling factors b_1 and b_2 yield the quite general relation $\boldsymbol{\mathcal{R}}_{b_2}\boldsymbol{\mathcal{R}}_{b_1} = \boldsymbol{\mathcal{R}}_{b_2 b_1}$, which essentially defines a unitary *semigroup* of transformations. the formal algebraic definition (MacLane and Birkhoff, 1967) of a unitary semigroup (or "monoid") is a set M of elements, $u, v, w, x, \ldots$ with a binary operation, $xy = w \in M$, which is associative, so $v(wx) = (vw)x$, and has a unit u, obeying $ux = xu = x$ (for all $x \in M$). In RG theory, the unit transformation corresponds simply to $b = 1$.

It is more fruitful to *iterate* the transformation so obtaining a sequence, $\overline{\mathcal{H}}^{(l)}$, of renormalized Hamiltonians, namely,

$$\overline{\mathcal{H}}^{(l)} = \boldsymbol{\mathcal{R}}_b \overline{\mathcal{H}}^{(l-1)} = \boldsymbol{\mathcal{R}}_{b^l} \overline{\mathcal{H}}. \tag{8.2}$$

Hille (Hille, 1948) and Riesz and Sz.-Nagy (Riesz and Sz.-Nagy, 1955) describe semigroups within a continuum, functional analysis context and discuss the existence of an infinitesimal generator when the flow parameter l is defined for continuous values $l \geq 0$. One may regard

$$l = \log_b(|\mathbf{r}'|/|\mathbf{r}|), \tag{8.3}$$

as measuring, logarithmically, the scale on which the system is being described; but note that, in general, the form of the Hamiltonian is also changing as the "scale" is changed or l increases. Thus a partially renormalized Hamiltonian can be expected to take on a more-or-less generic, mesoscopic form: Hence it represents an appropriate candidate to give meaning to a Landau-Ginzburg or, now, LGW effective Hamiltonian.

It is also worth mentioning that by letting $b \to 1^+$, one can derive a *differential* or continuous RG flow and rewrite the recursion relation (8.2) as

$$\frac{d}{dl}\overline{\mathcal{H}} = \boldsymbol{\mathcal{B}}\overline{\mathcal{H}}. \tag{8.4}$$

In this form the RG semigroup can typically be extended to an Abelian group (MacLane and Birkhoff, 1967). But as already stressed this fact plays a negligible role. Such continuous flows are illustrated in Fig. 2. [2]

The recursive application of an RG transformation $\boldsymbol{\mathcal{R}}_b$ induces a *flow* in the space of Hamiltonians, $\mathbb{H}$. Then one observes that "sensible", "reasonable", or, better, "well-designed" RG transformations are *smooth*, so that points in the original physical manifold, $\mathcal{H}^{(0)}$, that are close, say in temperature, remain so in $\mathcal{H}^{(1)}$, i.e., under renormalization, and likewise as the flow parameter l increases, in $\mathcal{H}^{(l)}$.

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[1] As opposed to the *momentum-shell* RG (Wilson and Fisher, 1972).

[2] If it happens that $\overline{\mathcal{H}}$ can be represented, in general only approximately, by a single coupling constant, say, g, then $\boldsymbol{\mathcal{B}}$ reduces to the so-called beta-function $\beta(g)$ of QFT.

Thanks to the smoothness of the RG transformation, if one knows the free energy $f_l \equiv f[\mathcal{H}^{(l)}]$ at the l-th stage of renormalization, then one knows the original free energy $f[\mathcal{H}]$ and its critical behavior: Explicitly one has

$$f(K) \equiv f[\overline{\mathcal{H}}] = b^{-dl} f[\overline{\mathcal{H}}^{(l)}] \equiv b^{-dl} f_l(K^{(l)}). \tag{8.5}$$

Furthermore, the smoothness implies that all the universal critical properties are preserved under renormalization. Similarly one finds (Fisher, 1983; Wilson, 1971a; Wilson and Kogut, 1974) that the critical point of $\overline{\mathcal{H}}^{(0)} \equiv \overline{\mathcal{H}}$ maps on to that of $\overline{\mathcal{H}}^{(1)} \equiv \overline{\mathcal{H}}'$, and so on, as illustrated by the flow lines in Fig. 2. Thus it is instructive to follow the *critical*

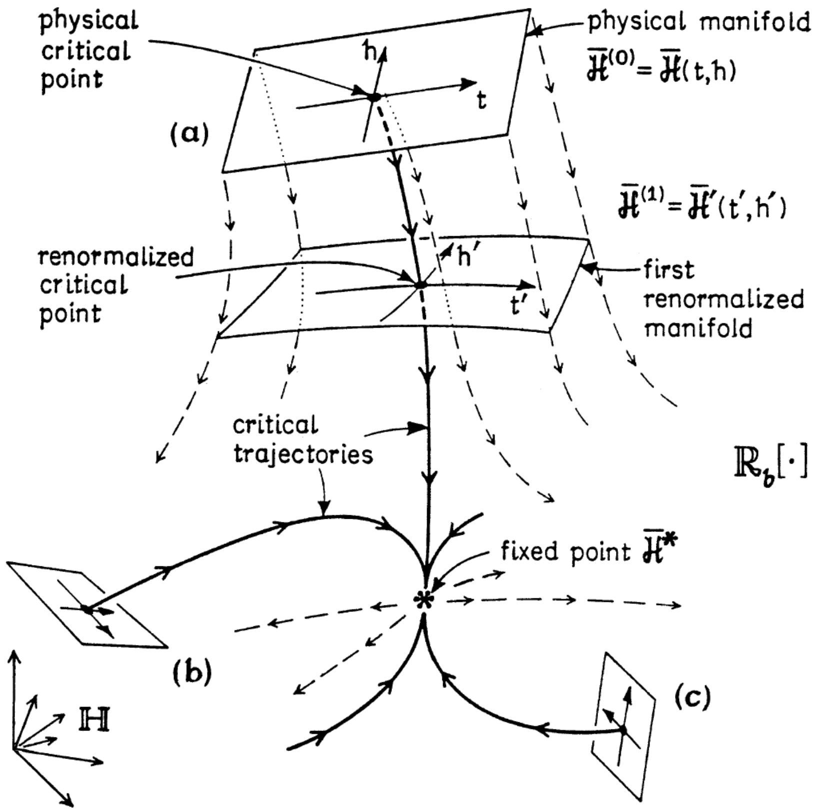

FIG. 2 A depiction of the space of Hamiltonians $\mathbb{H}$ showing initial or physical manifolds, $K = \{t, h\}$ with $t = (T - T_c)/T_c$ and T_c the critical temperature, [labelled (a), (b), ...] and the flows induced by repeated application of a discrete RG transformation $\mathcal{R}_b$ with a spatial rescaling factor b (or induced by a corresponding continuous or differential RG). Critical trajectories are shown bold: They all terminate, in the region of $\mathbb{H}$ shown here, at a fixed point $\overline{\mathcal{H}}^*$. The full space contains, in general, other nontrivial, critical fixed points, describing multicritical points and distinct critical-point universality classes; in addition, trivial fixed points, including high-temperature "sinks" with no outflowing or relevant trajectories, typically appear. Lines of fixed points and other more complex structures may arise and, indeed, play a crucial role in certain problems. [After Ref. (Fisher, 1983)]

trajectories in $\mathbb{H}$, i.e., those RG flow lines that emanate from a physical critical point. In principle, the topology of these trajectories could be enormously complicated and even chaotic: In practice, however, for a well-designed or "apt" RG transformation, one most frequently finds that the critical flows terminate, or, more accurately, come to an asymptotic halt, at a *fixed point* $\mathcal{H}^*$, of the RG: See Fig. 2. Such a fixed point is defined simply by

$$\mathcal{R}_b \overline{\mathcal{H}}^* = \overline{\mathcal{H}}^* \quad \text{or} \quad \mathcal{B}\overline{\mathcal{H}}^* = 0. \tag{8.6}$$

One then searches for fixed-point solutions.

Why are the fixed points so important? Some, in fact, are *not*, being merely *trivial*, corresponding to *no interactions* or to all spins frozen, etc. But the *nontrivial* fixed points represent critical states; furthermore, the nature of their criticality, and of the free energy in their neighborhood, must, as explained, be *identical* to that of all those distinct

Hamiltonians whose critical trajectories converge to the same fixed point. In other words, a particular fixed point defines a *universality class* [3] of critical behavior which "governs" or "attracts" all those systems whose critical points eventually map onto it: See Fig. 2.

Here, then we at last have the natural explanation of universality: Systems of quite different physical character may, nevertheless, belong to the domain of attraction of the same fixed point $\overline{\mathcal{H}}^*$ in $\mathbb{H}$. The distinct sets of inflowing trajectories reflect their varying physical content of associated irrelevant variables and the corresponding nonuniversal rates of approach to the asymptotic power laws dicated by $\mathcal{H}^*$.

From each critical fixed point, there flow at least two "unstable" or outgoing trajectories. These correspond to one or more relevant variables, specifically, for the case illustrated in Fig. 2, to the temperature or thermal field, $t = (T - T_c)/T_c$, with T_c the critical temperature, and the magnetic or ordering field, h.If there are further relevant trajectories then one can expect crossover to different critical behavior. In the space $\mathbb{H}$, such trajectories will then typically lead to distinct fixed points describing (in general) completely new universality classes. A skeptical reader may ask: "But what if no fixed points are found?" This can well mean, as it has frequently meant in the past, simply that the chosen RG transformation was poorly designed or "not apt". On the other hand, a fixed point represents only the simplest kind of asymptotic flow behavior: Other types of asymptotic flow may well be identified and translated into physical terms.

But what about *power laws* and *scaling*?

The smoothness of a well-designed RG transformation means that it can always be expanded locally, to at least some degree, in a Taylor series (Fisher, 1974b; Kadanoff, 1976; Wegner, 1972a,b, 1976; Wilson, 1971a; Wilson and Kogut, 1974). It is worth stressing that it is this very property that fails for free energies in a critical region: To regain this ability, the large space of Hamiltonians is crucial. Near a fixed point satisfying Eq. (8.5) we can, therefore, rather generally expect to be able to *linearize* by writing

$$\boldsymbol{\mathcal{R}}_b[\overline{\mathcal{H}}^* + g\mathcal{Q}] = \overline{\mathcal{H}}^* + g\boldsymbol{\mathcal{L}}_b\mathcal{Q} + o(g), \tag{8.7}$$

as $g \to 0$, or in differential form,

$$\frac{d}{dl}(\overline{\mathcal{H}}^* + g\mathcal{Q}) = g\boldsymbol{\mathcal{B}}\mathcal{Q} + o(g). \tag{8.8}$$

Now $\boldsymbol{\mathcal{L}}_b$ and $\boldsymbol{\mathcal{B}}$ are *linear operators* (albeit acting in a large space $\mathbb{H}$). As such we can seek eigenvalues and corresponding "eigenoperators", say $\mathcal{Q}_k$ (which will be "partial Hamiltonians"). Thus, we may write

$$\boldsymbol{\mathcal{L}}_b\mathcal{Q}_k = \Lambda_k(b)\mathcal{Q}_k \quad \text{or} \quad \boldsymbol{\mathcal{B}}\mathcal{Q}_k = \lambda_k\mathcal{Q}_k, \tag{8.9}$$

where, in fact, (by the semigroup property) the eigenvalues must be related by $\Lambda_k(b) = b^{\lambda_k}$. As in any such linear problem, knowing the spectrum of eigenvalues and eigenoperators or, at least, its dominant parts, tells one much of what one needs to know. Reasonably, the $\mathcal{Q}_k$ should form a basis for a general expansion

$$\overline{\mathcal{H}} \cong \overline{\mathcal{H}}^* + \sum_{k\geq 1} g_k\mathcal{Q}_k. \tag{8.10}$$

Physically, the expansion coefficient g_k ($\equiv g_k^{(0)}$) then represents the thermodynamic field (reduced, as always, by the factor $1/k_BT$) conjugate to the "critical operator" $\mathcal{Q}_k$ which, in turn, will often be close to some combination of *local* operators. Indeed, in a characteristic critical-point problem one finds two *relevant operators*, say $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with $\lambda_1, \lambda_2 > 0$. Invariably, one of these operators can, say by its symmetry, be identified with the local energy density, $\mathcal{Q}_1 \cong \mathcal{E}$, so that $g_1 \cong t$ is the thermal field; the second then characterizes the order parameter, $\mathcal{Q}_2 \cong \Psi$ with field $g_2 \cong h$. Under renormalization each g_k varies simply as $g_k^{(l)} \approx b^{\lambda_k l} g_k^{(0)}$.

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[3] This retrospective statement may, perhaps, warrant further comment. First, the terms "universal" and "universality class" came into common usage only after 1974 when the concept of various types of RG fixed point had been well recognized (see Fisher Ref. (Fisher, 1974b)). Kadanoff (Kadanoff, 1976) deserves credit not only for introducing and popularizing the terms but especially for emphasizing, refining, and extending the concepts. On the other hand, Domb's (Domb, 1960) review made clear that all (short-range) Ising models should have the same critical exponents irrespective of lattice structure but depending strongly on dimensionality. The excluded-volume problem for polymers was known to have closely related but distinct critical exponents from the Ising model, depending similarly on dimensionality but not lattice structure (Fisher and Sykes, 1959). And, as regards the Heisenberg model, which possesses what we would now say is an ($n = 3$)-component vector or $O(3)$ order parameter, there were strong hints that the exponents were again different (Domb and Sykes, 1962; Rushbrooke and Wood, 1958). On the experimental front matters might, possibly be viewed as less clear-cut: Indeed, for ferromagnets, nonclassical exponents were unambiguously revealed only in 1964 by Kouvel and Fisher (Kouvel and Fisher, 1964). However, a striking experiment by Heller and Benedek (Heller and Benedek, 1962) had already shown that the order parameter of the antiferromagnet MnF_2, namely, the sublattice magnetization $M_0^\dagger(T)$, vanishes as $|t|^\beta$ with $\beta = 0.33_5$. Furthermore, for fluids, the work of the Dutch school under Michels and the famous analysis of coexistence curves by Guggenheim (Guggenheim, 1949) allowed little doubt, see Rowlinson book (Rowlinson, 1959), Chap. 3, especially, pp. 91-95 that all reasonably simple atomic and molecular fluids displayed the same but nonclassical critical exponents with $\beta \simeq \frac{1}{3}$: And, also well before 1960, Widom and Rice (Widom and Rice, 1955) had analyzed the critical isotherms of a number of simple fluids and concluded that the corresponding critical exponent δ (see, e.g., Ref. (Fisher, 1967b)) took a value around 4.2 in place of the van der Waals value $\delta = 3$. In addition, evidence was in hand showing that the consolute point in binary fluid mixtures was similar (see Rowlinson book (Rowlinson, 1959), pp. 165-166).

Finally, one examines the flow equation (8.5) for the free energy. The essential point is that the degree of renormalization, b^l, can be chosen as large as one wishes. When $t \to 0$, i.e., in the critical region which it is our aim to understand, a good choice proves to be $b^l = 1/|t|^{1/\lambda_1}$, which clearly diverges at ∞. One then finds that Eq. (8.5) leads to the following *basic scaling relation*

$$f_s(t, h, \ldots, g_j, \ldots) \approx |t|^{2-\alpha} \mathcal{F}\left(\frac{h}{|t|^{\Delta}}, \ldots, \frac{g_j}{|t|^{\phi_j}}, \ldots\right), \tag{8.11}$$

where f_s is the "singular part" of the free energy found by subtracting from the free energy all the analytic terms. α is the specific heat exponent introduced while the exponent, Δ, which determines how h scales with t, is given by

$$\Delta = \beta + \gamma, \tag{8.12}$$

Widom observed, incidentally, that the classical theories themselves obey scaling: One then has $\alpha = 0, \Delta = 1\frac{1}{2}, \phi = -\frac{1}{2}$. The exponent, ϕ, did not appear in the original critical-point scaling formulations (Domb and Hunter, 1965; Fisher, 1967b; Kadanoff, 1966; Patashinskii and Pokrovskii, 1966; Stanley, 1971; Widom, 1965a,b); neither did the argument $g/|t|^{\phi}$ appear in the *scaling function* $\mathcal{F}$. It is really only with the appreciation of RG theory that we know that such a dependence should in general be present and, indeed, that a full spectrum $\{\phi_j\}$ of such higher-order exponents with $\phi \equiv \phi_1 > \phi_2 > \phi_3 > \ldots$ must normally appear (Fisher, 1974a; Wilson, 1971a).

Eq. (8.11) is the essential result. Recall, for example, that: (i) it very generally *implies* the thermodynamic exponent relation Eq. (5.10) connecting α, β, and γ (since the derivative of the free energy with respect to h is proportional to minus the magnetization); and (ii) since all leading exponents are determined entirely by the two exponents α and Δ $(= \beta + \gamma)$, it predicts similar exponent relations for any other exponents one might define, such as δ specified on the critical isotherm by $H \sim M^{\delta}$. Beyond that, (iii) if one fixes P (or g) and similar parameters and observes the free energy or, in practice, the *equation of state*, the data one collects amount to describing a function, say $M(T, H)$, of *two variables*. Typically this would be displayed as sets of *isotherms*: i.e., many plots of M vs. H at various closely spaced, fixed values of T near T_c . But according to the scaling law Eq. (8.11) if one plots the *scaled variables* $f_s/|t|^{2-\alpha}$ or $M/|t|^{\beta}$ vs. the scaled field $h/|t|^{\Delta}$, for appropriately chosen exponents and critical temperature T_c, one should find that all these data "collapse" (in Stanley's (Stanley, 1971) picturesque terminology) onto a single curve, which then just represents the scaling function $x = \mathcal{F}(y)$ itself. This collapse is some times also called *law of corresponding states* (see for instance section 4.1 in Ref. (Hansen and McDonald, 1990)).

Now, however, the critical exponents can be expressed directly in terms of the RG eigenexponents λ_k (for the fixed point in question). Specifically one finds

$$2 - \alpha = \frac{d}{\lambda_1}, \quad \Delta = \frac{\lambda_2}{\lambda_1}, \quad \phi_j = \frac{\lambda_j}{\lambda_1}, \quad \nu = \frac{1}{\lambda_1}. \tag{8.13}$$

Then, the sign of a given ϕ_j and, hence, of the corresponding λ_j determines the *relevance* (for $\lambda_j > 0$), *marginality* (for $\lambda_j = 0$), or *irrelevance* (for $\lambda_j < 0$) of the corresponding critical operator $\mathcal{Q}_j$ (or "perturbation") and of its conjugate field g_j: This field might, but for most values of j will not, be under direct experimental control. The first and last of the equations (8.13) yield the *hyperscaling relation*: $d\nu = 2 - \alpha$ which explicitly involve the spatial dimensionality (Fisher, 1974a). This relation holds *exactly* for the $d = 2$ Ising model and also for all other exactly soluble models when $d < 4$ (Baxter, 1982; Fisher, 1983). [4]

When a coupling constant g is irrelevant then $z = g/|t|^{\phi} \to 0$ on approaching the critical point. Consequently, $\mathcal{F}(y, z)$ can be replaced simply by $\mathcal{F}(y, 0)$ which is a function of just a *single variable*. Furthermore, asymptotically when $T \to T_c$ we get the same function whatever the actual value of g. Clearly this is an example of *universality*. [5] Then one can, fairly generally, hope to expand the scaling function $\mathcal{F}(y, z)$ in powers of z and thereby obtain the so called *"correction-to-scaling" exponent* θ, which is also universal (for $d = 3$ Ising-type systems one finds $\theta \simeq 0.54$ (Zinn and Fisher, 1996)).

When a coupling constant g is relevant then when $t \to 0$ the scaled variable $g/|t|^{\phi}$ grows larger and larger. Two possibilities then arise: *Either* the critical point may be *destroyed* altogether. This is, in fact, the effect of the

[4] Unlike the previous exponent relations (all being independent of d) hyperscaling fails for the classical theories unless $d = 4$. And since one knows (rigorously for certain models) that the classical exponent values are valid for $d > 4$, it follows that hyperscaling cannot be generally valid. Thus something is certainly missing from Kadanoff's picture. Now, thanks to RG insights, we know that the breakdown of hyperscaling is to be understood via the second argument in the "fuller" scaling form Eq. (8.11): when d exceeds the appropriate borderline dimension, d_{uc}, a "dangerous irrelevant variable" appears and must be allowed for (see Fisher in Ref. (Gunton and Green, 1973) p. 66 where a "dangerous irrelevant variable" is characterized as a "hidden relevant variable" and Ref. (Fisher, 1983), appendix D). In essence one finds that the scaling function limit $\mathcal{F}(y, z \to 0, \ldots)$, previously accepted without question, is no longer well defined but, rather, diverges as a power of z: asymptotic scaling survives but $d^* \equiv (2 - \alpha)/\nu$ sticks at the value 4 for $d > d_{uc} = 4$.

[5] Note that T_c for example, will usually be a function of any irrelevant parameter such as g_j. This comes about because, in a full scaling formulation, the variables t, h, and $\{g_j\}$ appearing in Eq. (8.11) must be replaced by nonlinear scaling fields $t(t, h, \{g_j\}), h(t, h, \{g_j\})$, and $g_j(t, h, \{g_j\})$ which are smooth functions of t, h, and g_j (Fisher, 1983; Wegner, 1972a,b, 1976). By the same token it is usually advantageous to introduce a prefactor A_0 in Eq. (8.11) and "metrical factors" E_j in the arguments of $\mathcal{F}$ (see, e.g., Ref. (Fisher, 1983).

magnetic field, which must itself be regarded as a relevant perturbation since $\phi \equiv \Delta = \beta + \gamma > 0$. *Alternatively*, when z grows, the true, asymptotic critical behavior may *crossover* (Aharony, 1976; Fisher, 1974b) to a new, quite *distinct* universality class with different exponents and a new asymptotic scaling function, say, $\mathcal{F}_\infty(y)$. [6]

Whena coupling constant g is marginal then when $t \to 0$ this may lead to *logarithmic* modifications of the classical critical power laws (by factors diverging as $\ln|t|$ to various powers). The predicted logarithmic behavior has, in fact, been verified experimentally by Ahlers et al. (Ahlers *et al.*, 1975). In other cases, especially for $d = 2$, marginal variables lead to continuously variable exponents such as $\alpha(g)$, and to quite different thermal variation, like $\exp(A/|t|^\nu)$; such results have been checked both in exactly solved statistical mechanical models and in physical systems such as superfluid helium films (Kadanoff and Wegner, 1971; Nelson, 1983).

Because of the multifaceted character of matter physics these are rather different and more diverse than those aspects of RG theory of significance for QFT. When there are no marginal variables and the leas negative ϕ_j is larger than unity in magnitude, a simple scaling description will usually work well and the Kadanoff picture almost applies. When there are no relevant variables and only one or a few marginal variables, field-theoretic perturbative techniques of the Gell-Mann-Low (Gell-Mann and Low, 1954), Callan-Symanzik (Amit, 1978; Brézin *et al.*, 1976; Itzykson and Drouffe, 1989; Wilson, 1975) or so-called "parquet diagram" varieties (Larkin and Khmel'nitskii, 1969) may well suffice (assuming the dominating fixed point is sufficiently simple to be well understood). There may then be little incentive for specifically invoking general RG theory. This seems, more or less, to be the current situation in QFT and it applies also in certain physics of matter problems.

Within RG theory the general mechanism of universality is as follows: In a very large (generally infinitely large) space of Hamiltonians $\mathbb{H}$, parametrized by t, h, and all the g_j , there is a controlling critical point (a fixed point) about which each variable enters with a characteristic exponent. All systems with Hamiltonians differing only through the values of the g_j (within suitable bounds) will exhibit the same critical behavior determined by the same free-energy scaling function $\mathcal{F}$, dropping the irrelevant arguments. Different universality classes will be associated with different controlling critical points in the space of Hamiltonians with, once one recognizes the concept of RG flows, different "domains of attraction" under the flow. Indeed, the expectation of a general form of scaling is frequently the most important consequence of RG theory for the practising experimentalist or theorist.

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[6] Formally, one might write $\mathcal{F}_\infty(y) = \mathcal{F}(y, z \to z_\infty)$ where z_∞ is a critical value which could be ∞; but a more subtle relationship is generally required since the exponent α in the prefactor in Eq. (8.11) changes

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