Angular momentum & rotations

Riccardo Fantoni*

Università di Trieste,
Dipartimento di Fisica,
strada Costiera 11,
34151 Grignano (Trieste),
Italy

(Dated: June 28, 2017)

We want to define the angular momentum as the generator of the rotations in quantum mechanics.

CONTENTS

I. PRELIMINARIES

Consider the orthogonal transformation $\mathbf{q}' = \gamma(\mathbf{q})$ with γ a proper orthogonal matrix. This transformation can be interpreted in two ways:

(1) active rotation: rotating the system the physical points go from the position of coordinates $\mathbf{q}$ to the one of coordinates $\mathbf{q}'$;

(2) passive rotation: changing the reference frame the same point is described with two different coordinates.

What I' ll say next holds for both attitudes except when explicitly noted.

If ψ describes a state, let $\psi' = T\psi$ be the vector which describes the state of the rotated system or of the same system described in the rotated reference frame.

Wigner postulate the invariance of the transition probabilities, i.e.

$$\frac{|(\phi,\psi)|^2}{(\phi,\phi)(\psi,\psi)} = \frac{|(T\phi,T\psi)|^2}{(T\phi,T\phi)(T\psi,T\psi)} \quad . \tag{1.1}$$

Any set of transformations T with the inverse, and satisfying equation (1.1) is a group and is called group of symmetry. Since, up to this point, T is a general transformation, not necessarily linear, the transformation T can always be choosen so as to conserve the norm. Thus we will impose,

$$(T\psi, T\psi) = (\psi, \psi) \quad . \tag{1.2}$$

A change of the kind,

$$T \to T' \quad \text{such that} \quad T'\psi = e^{i\alpha(\psi)} T\psi \quad , \tag{1.3}$$

leaves equations (1.1) and (1.2) unaltered. Thus we can try to use this degree of freedom to reduce the operator T to a more conventional form. Wigner (E.P.Wigner: Group Theory, Academic Press (1959) pag.233) shows that is always possible to choose the phases in (1.3), in such a way to have T linear or antilinear (not both cases are realizable starting from a given transformation T).

* rfantoni@ts.infn.it

In the linear case equation (1.2) tells us that L (the name given to this linear operator) is isometric, i.e. $L^\dagger L = 1$. If moreover we assume that the image of Hilbert space $\mathcal{H}$ under L is the whole Hilbert space (which is always true if T has an inverse) then L is also unitary ($\forall g \in \mathcal{H}\ \exists f \in \mathcal{H} \mid g = Lf \Rightarrow LL^\dagger L = L \Rightarrow LL^\dagger g = g$).

Consider now the antilinear case. The definition of antilinear operator is,

$$A(\alpha\psi + \beta\phi) = \alpha^\star A\psi + \beta^\star A\phi \quad . \tag{1.4}$$

If $(A\psi, A\psi) = (\psi, \psi)$, then from definition (1.4) follows,

$$\begin{aligned} &(A(\alpha\psi + \beta\phi), A(\alpha\psi + \beta\phi)) = \\ &= |\alpha|^2(\psi,\psi) + \beta^\star\alpha(\phi,\psi) + \beta\alpha^\star(\psi,\phi) + |\beta|^2(\phi,\phi) \\ &= |\alpha|^2(A\psi,A\psi) + \beta^\star\alpha(A\phi,A\psi) + \beta\alpha^\star(A\psi,A\phi) + |\beta|^2(A\phi,A\phi) \quad , \end{aligned}$$

and from the arbitrariness of α and β we get $(A\psi, A\phi) = (\psi, \phi)$ which defines an antisimmetric operator. The complex number $(\psi, A\phi)^\star = (A\phi, \psi)$ is linearly dependent on ϕ and can then be written using Riesz theorem as (ζ, ϕ), i.e.

$$(\psi, A\phi) = (\phi, \zeta) \quad ,$$

with ζ antilinearly dependent on ψ. So we can introduce the antilinear operator $A^\dagger$, called the adjoint of A and defined by,

$$A^\dagger\psi = \zeta \quad .$$

The invariance of the norm of ψ tells us that A is isometric, i.e. $A^\dagger A = 1$, and again the hypothesis that the image of $\mathcal{H}$ under A is the whole $\mathcal{H}$ tells us that $AA^\dagger = 1$ and A is called antiunitary.

This considerations hold for all symmetry operations. I want to show now that all symmetry operations that don' t involve time reversal and commute with the Hamiltonian H, have to be unitary in order to be consistent with the superposition principle.

Consider the superposition of two eigenstates of the energy ψ_1 and ψ_2 with different eigenvalues E_1 and E_2. Assume the symmetry transformation to be antiunitary by absurd. The state $\alpha_1\psi_1 + \alpha_2\psi_2$ at time 0, evolves at time t into

$$\alpha_1 e^{-iE_1t/\hbar}\psi_1 + \alpha_2 e^{-iE_2t/\hbar}\psi_2 \quad . \tag{1.5}$$

Since we assumed $[A, H] = 0$ (this is always true for passive rotations), the transformed state $\alpha_1^\star A\psi_1 + \alpha_2^\star A\psi_2$ at time 0, evolves at time t into

$$\alpha_1^\star e^{-iE_1t/\hbar}A\psi_1 + \alpha_2 e^{-iE_2t/\hbar}A\psi_2 \quad . \tag{1.6}$$

Now transforming state (1.5) under A we have to find state (1.6). That is, the vector

$$\alpha_1^\star e^{iE_1t/\hbar}A\psi_1 + \alpha_2 e^{iE_2t/\hbar}A\psi_2 \quad ,$$

can differ from the state

$$\alpha_1^\star e^{-iE_1t/\hbar}A\psi_1 + \alpha_2 e^{-iE_2t/\hbar}A\psi_2 \quad ,$$

only by a phase factor. But since the two state $A\psi_1$ and $A\psi_2$ are orthogonal and $E_1 \neq E_2$ this cannot be valid $\forall t$. Thus assuming an antiunitary transformation lead to a contradiction.

Consider a group of symmetry transformations representable through unitary operators. Let γ_1 and γ_2 be represented by $U(\gamma_1)$ and $U(\gamma_2)$, then $\gamma_1\gamma_2$ will be represented by $U(\gamma_1\gamma_2)$. Acting first with γ_1 and then γ_2 is physically equivalent to acting with $\gamma_2\gamma_1$. This means that,

$$U(\gamma_2\gamma_1)\psi = \alpha(\gamma_1, \gamma_2, \psi)U(\gamma_2)U(\gamma_1)\psi \quad , \tag{1.7}$$

where α is a phase factor. A corrispondence $\gamma \to U(\gamma)$ satisfying (1.7) is called a projective representation of the symmetry group.

A simple argument shows that, due to the unitariety of U, α cannot depend on ψ. Consider two unitary operators U and V such that $\forall\psi$, $U\psi = \alpha V\psi$ with $\alpha = \alpha(\psi)$. Let $K = V^\dagger U$. Given ψ_1 and ψ_2 linearly independent, $K\psi_1 = \alpha_1\psi_1$, $K\psi_2 = \alpha_2\psi_2$, and

$$\begin{aligned} K(a_1\psi_1 + a_2\psi_2) &= \alpha_1 a_1\psi_1 + \alpha_2 a_2\psi_2 = \\ \alpha_3(a_1\psi_1 + a_2\psi_2) &= \alpha_3 a_1\psi_1 + \alpha_3 a_2\psi_2 \quad . \end{aligned} \tag{1.8}$$

Since ψ_1 and ψ_2 are independent we must have $\alpha_3 = \alpha_1$ and $\alpha_3 = \alpha_2$, i.e. $\alpha_1 = \alpha_2 =$constant.

One can easily show (V.Bargmann, Ann. of Math. **59**, 1, (1952)) that given a projective representation (i.e. satisfying (1.7)) continuous in a neighborhood of the identity one can make a phase transformation on the U's such that $U \to \omega(U)U$, with ω phase factor, in such a way that in a neighborhood of the identity the representation remains continuous and becomes a genuine representation (i.e. satisfy (1.7) with $\alpha = 1$). In order for this to be possible is crucial the property of a neighborhood of the identity, of being simply connected. The same doesn' t hold in general, for the whole representation. For $SO(3)$ for example, which is not a simply connected group, in general is not possible to redefine the phases in order to have a continuous representation with $\alpha = 1$ in (1.7).

Let' s now specialize our considerations to the group of rotations $SO(3)$. The main fact that distinguishes this group from the translations is its non commutativity. Given two infinitesimal rotations characterized by the antisymmetric transformations α and β, we want to calculate the commutator of the two transformations generated by α and β, i.e. $\exp(-\beta)\exp(-\alpha)\ \exp(\beta)\exp(\alpha)$ keeping up to second order terms. Using twice Campbell-Baker-Hausdorff relation, $\exp(A)\exp(B) = \exp(A + B + [A,B]/2 + O(3))$, we get

$$e^{-\beta-\alpha+[\beta,\alpha]/2+O(3)} e^{\beta+\alpha+[\beta,\alpha]/2+O(3)} = e^{[\beta,\alpha]+O(3)} \quad . \tag{1.9}$$

According to Wigner theorem given a rotation it can always be represented in the Hilbert space using a unitary transformation. Indicate with $\exp(-ir(\alpha)/\hbar)$ the unitary transformation relative to rotation α and with $\exp(-ir(\beta)/\hbar)$ the one relative to rotation β, where r are autoadjoint operators. If we take the commutator of these two transformation, using the same procedure used to get (1.9) and imposing (1.7) we get,

$$e^{[-ir(\beta)/\hbar, -ir(\alpha)/\hbar]} = e^{-ir([\beta,\alpha])/\hbar + i\phi(\beta,\alpha)} \quad , \tag{1.10}$$

where as previously shown, ϕ can depend on α and β. We have already said that in a neighborhood of the identity we can choose the phases so that

$$U(\alpha)U(\beta) = U(\alpha\beta) \quad .$$

With this choice of zero phase in (1.10) we get

$$[r(\alpha), r(\beta)] = i\hbar r([\alpha,\beta]) \quad . \tag{1.11}$$

Let' s specify now the transformations α, β, ... to the infinitesimal rotations around the coordinated axes.

$$\alpha_1 = \epsilon_1 A_1, \quad \alpha_2 = \epsilon_2 A_2, \quad \alpha_3 = \epsilon_3 A_3 \quad .$$

with A_1, A_2, A_3 given by

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad .$$

For example α_3 give the following infinitesimal transformation

$$\begin{cases} q_1' = q_1 - \epsilon_3 q_2 \\ q_2' = q_2 + \epsilon_3 q_1 \\ q_3' = q_3 \end{cases}$$

The three A matrices satisfy the following commutation relations

$$[A_i, A_j] = \varepsilon_{i,j,k} A_k \quad .$$

Taking for simplicity

$$r(\epsilon_j A_j) = \epsilon_j A_j \qquad \text{(without summing over j)} \quad , \tag{1.12}$$

and using (1.11) we get

$$[A_i, A_j] = i\hbar\varepsilon_{i,j,k} A_k \quad , \tag{1.13}$$

which are the commutation relations of the orbital angular momentum. This is a general statement which holds without having to specify the nature of the vector which we are tranforming.

II. ROTATIONS OF WAVES FUNCTIONS

Assume that the state is represented by the wave function $\psi(\mathbf{q})$. The easiest and more natural way to transform the wave function under rotations is obtained by imposing the invariance in value of the wave function, i.e.

$$\psi'(\mathbf{q}') = \psi'(\gamma_1(\mathbf{q})) = \psi(\mathbf{q}) \quad \text{i.e.} \quad \psi'(\mathbf{q}) = \psi(\gamma_1^{-1}(\mathbf{q})) \quad . \tag{2.1}$$

For two successive transformations γ_1 and then γ_2, we have

$$\psi''(\mathbf{q}) = \psi'(\gamma_2^{-1}(\mathbf{q})) = \psi(\gamma_1^{-1}\gamma_2^{-1}(\mathbf{q})) = \psi((\gamma_2\gamma_1)^{-1}(\mathbf{q})) \quad .$$

Since the Jacobian of an orthogonal transformation is 1, then

$$\int \psi^\star(\gamma^{-1}(\mathbf{q}))\phi(\gamma^{-1}(\mathbf{q}))d\mathbf{q} = \int \psi^\star(\mathbf{q})\phi(\mathbf{q})d\mathbf{q} \quad .$$

This means that the transformation (2.1) is unitary. We have then $\psi'(\mathbf{q}) = U(\gamma_1)\psi(\mathbf{q})$ and $U(\gamma_2\gamma_1) = U(\gamma_2)U(\gamma_1)$ without any additional phase.

We have thus shown that the transformation in value of the wave function, completely realize the plan of obtaining for SO(3) a true representations also for finite transformations.

III. ROTATIONS OF SPINORS

Consider the bidimensional Hilbert space made of the bi-complexes $\begin{pmatrix} a \\ b \end{pmatrix}$ (the spinors), and the following linear hermitian operators,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

called the Pauli matrixes. We can easily verify that

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{i,j,k}\sigma_k \quad .$$

Thus taking $s_i = \hbar\sigma_i/2$ we solve the problem of finding three operators s_i satisfying the commutation relations for the angular momentum (1.13). From the commutation relations and the additional property $\sigma_i^2 = 1$, one can easily verify that the σ_i satisfy the Clifford algebra, namely

$$\{\sigma_i, \sigma_j\} = 2\delta_{i,j} \quad ,$$

where $\{,\}$ denote the anticommutator.

According to equation (1.12) the infinitesimal rotation of an angle ϵ around the axis $\mathbf{n}$ is given by

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = (1 - i\mathbf{s}\cdot\mathbf{n}\epsilon/\hbar)\begin{pmatrix} a \\ b \end{pmatrix} = (1 - i\boldsymbol{\sigma}\cdot\mathbf{n}\epsilon/2)\begin{pmatrix} a \\ b \end{pmatrix} \quad ,$$

and since $(\boldsymbol{\sigma}\cdot\mathbf{n})^2 = 1$, the finite rotation of an angle ϕ around $\mathbf{n}$, is given by

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = e^{-i\boldsymbol{\sigma}\cdot\mathbf{n}\phi/2}\begin{pmatrix} a \\ b \end{pmatrix} = (\cos(\phi/2) - i\boldsymbol{\sigma}\cdot\mathbf{n}\sin(\phi/2))\begin{pmatrix} a \\ b \end{pmatrix} \quad .$$

For a rotation af 2π around any axis one has $\begin{pmatrix} a' \\ b' \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$; this is not against the physical interpretation of the state vector.

The 2×2 matrices,

$$U = \cos(\phi/2) - i\boldsymbol{\sigma}\cdot\mathbf{n}\sin(\phi/2) \quad , \tag{3.1}$$

are the whole and only elements of the group SU(2), i.e. the group of unitary transformations with determinant equal to 1 in two dimensions.

This can be shown for example introducing the 2×2 identity matrix σ_0 and writing the more general bidimensional matrix as $a\sigma_0 + \mathbf{b}\cdot\boldsymbol{\sigma}$. The determinant of this matrix is given by $a^2 - \mathbf{b}^2$. As immediately follows from the Clifford algebra the inverse of that unimodular matrix is $a\sigma_0 - \mathbf{b}\cdot\boldsymbol{\sigma}$. Now if we want the inverse to coincide with the adjoint, we must have that $a = a^\star$ and $\mathbf{b} = -\mathbf{b}^\star$. So the more general matrix of SU(2) can be written as

$$a\sigma_0 + i\mathbf{b}\cdot\boldsymbol{\sigma} \quad ,$$

with a and $\mathbf{b}$ reals and $a^2 + \mathbf{b}^2 = 1$. This means that SU(2) is in a bijective and continuous corrispondence with the points of a 4-dimensional sphere of radius 1, wich is a simply connected set. In the parametrization of equation (3.1) the angle can be choosen to be $0 \leq \phi \leq 2\pi$.

We now want to show that this corrispondence between the elements of SO(3) and the elements of SU(2) is a projective representation of the group SO(3), i.e. given two elements of SO(3), γ_1 and γ_2, the corrispondent elements of SU(2), $U(\gamma_1)$ and $U(\gamma_2)$ must be such that

$$U(\gamma_2\gamma_1) = \alpha(\gamma_2,\gamma_1)U(\gamma_2)U(\gamma_1) \quad ,$$

with $\alpha(\gamma_2,\gamma_1)$ a phase factor.

Given an element γ of SO(3), i.e. the rotation of an angle ϕ around an axis $\mathbf{n}$, this corresponds (modulo a sign) to the element $U(\gamma)$ of SU(2). Let' s start by showing the following relation

$$U^\dagger(\gamma)\boldsymbol{\sigma}U(\gamma) = \gamma(\boldsymbol{\sigma}) \quad .$$

Under a rotation of an angle ϕ around $\mathbf{n}$ one has

$$\mathbf{q} \to \mathbf{q}' = \gamma(\mathbf{q}) = \mathbf{n}(\mathbf{q}\cdot\mathbf{n}) + \cos(\phi)(\mathbf{q} - \mathbf{n}(\mathbf{q}\cdot\mathbf{n})) + \sin(\phi)\mathbf{n}\wedge\mathbf{q} \quad .$$

Using the relation $(\boldsymbol{\sigma}\cdot\mathbf{n})\sigma_k(\boldsymbol{\sigma}\cdot\mathbf{n}) = -\sigma_k + 2n_k\boldsymbol{\sigma}\cdot\mathbf{n}$ (that follows from Clifford algebra) one finds

$$(\cos(\phi/2) + i\boldsymbol{\sigma}\cdot\mathbf{n}\sin(\phi/2))\boldsymbol{\sigma}(\cos(\phi/2) - i\boldsymbol{\sigma}\cdot\mathbf{n}\sin(\phi/2)) = \gamma(\boldsymbol{\sigma}) \quad .$$

Given now two elements of SO(3), γ_1 and γ_2 and their product $\gamma_2\gamma_1$ we have

$$\begin{gathered} U^\dagger(\gamma_1)U^\dagger(\gamma_2)\boldsymbol{\sigma}U(\gamma_2)U(\gamma_1) = U^\dagger(\gamma_1)\gamma_2(\boldsymbol{\sigma})U(\gamma_1) = \\ \gamma_2\gamma_1(\boldsymbol{\sigma}) = U^\dagger(\gamma_2\gamma_1)\boldsymbol{\sigma}U(\gamma_2\gamma_1) \quad . \end{gathered}$$

This means that the unitary operator $V = U(\gamma_2\gamma_1)U^\dagger(\gamma_1)U^\dagger(\gamma_2)$ is such that

$$V^\dagger\boldsymbol{\sigma}V = \boldsymbol{\sigma} \quad , \tag{3.2}$$

or

$$\boldsymbol{\sigma}^\dagger V = V\boldsymbol{\sigma} \quad . \tag{3.3}$$

Since V is an element of SU(2) this imply $V = 1$ or $V = -1$. We can then say that (3.1) give a projective representation of SO(3), i.e. equation (1.7) holds with $\alpha = \pm 1$. This tells us also that if we have a sequence of SO(3) transformations with product the identity, under the product of the corrispondent transformations of SU(2) the spinor can only go into itself or change sign. Viceversa given an element U of SU(2) we can write

$$U^\dagger\sigma_j U = \Gamma_{ji}\sigma_i \quad , \tag{3.4}$$

infact the trace of the left hand side is zero. Since the left hand side is an hermitian operator we have that the elements $\Gamma_{j,i}$ are reals. Making the product of two of these relations and taking the trace we get

$$\delta_{jk} = \Gamma_{ji}\Gamma_{ki} \quad ,$$

which implies that $\Gamma_{j,i}$ are elements of the group O(3). If we now take the trace of $U^\dagger\sigma_1 UU^\dagger\sigma_2 UU^\dagger\sigma_3 U$ we get

$$2i = 2i\varepsilon_{ijk}\Gamma_{1i}\Gamma_{2j}\Gamma_{3k} \quad ,$$

i.e. $det(\Gamma) = 1$. U and $-U$ through (3.4) generate the same Γ. Viceversa if U and V generate the same Γ from equation (3.2,3.3) follows $U = \pm V$. Then we can say that to any element of SU(2) corrisponds an element of SO(3) while to any element of SO(3) correspond two elements of SU(2) given by $\pm U$. SU(2) is a simply connected group that is called the universal covering of O(3).