Sum Rules in Quantum Liquids

Riccardo Fantoni*

Università di Trieste, Dipartimento di Fisica, strada Costiera 11, 34151 Grignano (Trieste), Italy
(Dated: January 18, 2025)

We review the linear response theory in the horizontal quantum liquids framework spanning from Coulomb liquids to Atomic gases.

Keywords: Sum Rules, Quantum Liquids, Coulomb liquids, Atomic gases

I. INTRODUCTION

Linear response theory [1–3] is a well known framework to describe the approach to thermal equilibrium in response to an external perturbation acting on a many-body quantum fluid. We discuss two examples of quantum fluids: The Coulomb Liquid and an Atomic Gas.

II. LINEAR RESPONSE THEORY

Let us introduce the density linear response function $K(\mathbf{r} - \mathbf{r}', t - t')$ for a homogeneous fluid. Let us indicate with V_b the "bare" potential in vacuum.

The coupling of the fluid to the perturbing potential is described by the Hamiltonian

$$H'(t) = \int d\mathbf{r} \, \rho(\mathbf{r}) V_b(\mathbf{r}, t), \tag{2.1}$$

where $\rho(\mathbf{r})$ is the density operator (here we implicitly assume that the mean value of the density has been subtracted from $\rho(\mathbf{r})$). We will just consider the linear effect of this perturbation. The change in density is given by

$$\delta n(\mathbf{r}, t) = \langle \rho(\mathbf{r}) \rangle - \langle \rho(\mathbf{r}) \rangle_0 = \mathrm{tr}\{[w(t) - w_0]\rho(\mathbf{r})\}, \tag{2.2}$$

where tr denotes the trace, $w(t) = \int \psi^*(R, t)\psi(R, t) \, d^{3N}R$ is the perturbed density matrix whose unperturbed counterpart is $w_0 = \exp(-\beta H_0)/tr\{\exp(-\beta H_0)\}$, and $\beta = 1/k_B T$ with k_B the Boltzmann constant and T the absolute temperature. We are indicating with $\psi(R, t)$ the many-body wave function of the fluid with particles at positions $R = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ at time t. This satisfies to the Schrödinger equation

$$i\hbar \frac{\partial \psi(R, t)}{\partial t} = [H_0 + H'(t)]\psi(R, t), \tag{2.3}$$

where H is the Hamiltonian of the unperturbed fluid. Then the perturbed density matrix satisfies to

$$
\begin{aligned}
i\hbar \frac{\partial w(t)}{\partial t} &= [H_0 + H'(t), w(t)] \\
&\approx [H_0, w(t) - w_0] + [H'(t), w_0],
\end{aligned}
\tag{2.4}
$$

where $[A, B]$ denotes the commutator $AB - BA$ and in the last step we have linearized the effect of the perturbation and used $[H_0, w_0] = 0$. This equation is subject to the initial condition

$$\lim_{t \to -\infty} w(t) = w_0, \tag{2.5}$$

representing a state of thermal equilibrium.

The linearized equation (2.4) has the following solution

$$w(t) - w_0 = (i\hbar)^{-1} \int_{-\infty}^{t} dt' \, \exp\{-iH_0(t - t')/\hbar\}[H'(t'), w_0]\exp\{iH_0(t - t')/\hbar\}. \tag{2.6}$$

* riccardo.fantoni@scuola.istruzione.it

Inserting this result into Eq. (2.2) and using the cyclic invariance of the trace, $\text{tr}\{AB\} = \text{tr}\{BA\}$, we can write the desired result as follows

$$\delta n(\mathbf{r}, t) = (-i/\hbar) \int d\mathbf{r}' \int_{-\infty}^{t} dt' \, \langle [\rho(\mathbf{r}, t), \rho(\mathbf{r}', t')] \rangle_0 V_b(\mathbf{r}', t). \tag{2.7}$$

Again the angle parenthesis $\langle A \rangle_0 = \text{tr}\{w_0 A\}$ denotes the mean value on the equilibrium state and $\rho(\mathbf{r}, t)$ is the Heisenberg operator

$$\rho(\mathbf{r}, t) = \exp(iH_0 t/\hbar)\rho(\mathbf{r})\exp(-iH_0 t/\hbar). \tag{2.8}$$

So

$$K(\mathbf{r} - \mathbf{r}', t - t') = (-i/\hbar)\theta(t - t')\langle [\rho(\mathbf{r}, t), \rho(\mathbf{r}', t')] \rangle_0. \tag{2.9}$$

This result clearly embodies the causality property through the Heaviside step function θ.

Introducing the notation

$$\chi''(k, t - t') = (1/2\hbar) \int d(\mathbf{r} - \mathbf{r}') \, \exp[-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')]\langle [\rho(\mathbf{r}, t), \rho(\mathbf{r}', t')] \rangle_0, \tag{2.10}$$

we see, from Eq. (2.9) that the Fourier transform of K is the convolution integral of the Fourier transform of $\chi''(k, t)$, that we will indicate with $\chi''(k, \omega)$, and of the Heaviside step function, that is equal to $i/(\omega + i\eta)$ with η a small positive quantity. We can then write the space-time Fourier transform of K like so

$$\chi(k, \omega) = -\int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \, \chi''(k, \omega')/(\omega - \omega' + i\eta). \tag{2.11}$$

Using the rule $(\omega + i\eta)^{-1} = P(1/\omega) - i\pi\delta(\omega)$, where P denotes the Cauchy principal part, this can be written like so

$$\chi(k, \omega) = -P\int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \, \chi''(k, \omega')/(\omega - \omega') + i\chi''(k, \omega). \tag{2.12}$$

Since $\chi''(k, t)$ is written in terms of the commutator of Hermitian operators it can be readily shown that $\chi''(k, \omega)$ must be real. So we can write

$$\text{Im}\chi(k, \omega) = \chi''(k, \omega). \tag{2.13}$$

III.   FLUCTUATION-DISSIPATION THEOREM

We now worry about the relationship between the density response function and the van Hove dynamic response $S(\mathbf{k}, \omega)$. Let us define the autocorrelation density function as

$$G(\mathbf{r} - \mathbf{r}', t - t') = \frac{1}{n}\langle \rho(\mathbf{r}, t)\rho(\mathbf{r}', t') \rangle_0, \tag{3.1}$$

whose space-time Fourier transform is $S(\mathbf{k}, \omega)$. The connection between G e K that gush from Eq. (2.9) can be rewritten in Fourier transform like so

$$\chi(\mathbf{k}, \omega) = (n/\hbar) \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} [S(\mathbf{k}, \omega) - S(-\mathbf{k}, -\omega)]/(\omega - \omega' + i\eta). \tag{3.2}$$

This has the same form of Eq. (2.11) so that

$$\text{Im}\chi(\mathbf{k}, \omega) = (-n/2\hbar)[S(\mathbf{k}, \omega) - S(-\mathbf{k}, -\omega)]. \tag{3.3}$$

For a fluid in thermodynamic equilibrium we must have

$$S(-\mathbf{k}, -\omega) = \exp(-\hbar\beta\omega)S(\mathbf{k}, \omega). \tag{3.4}$$

In order to prove this property we observe that its inverse space-time Fourier transform reads

$$G(-\mathbf{r}, -t) = \exp\left(-i\hbar\beta\frac{\partial}{\partial t}\right) G(\mathbf{r}, t), \tag{3.5}$$

since under time Fourier transform $\partial/\partial t \to -i\omega$. But Eq. (3.5) can readily be proven through the following steps (where, once again we use the cyclic invariance of the trace and the definition of the Heisenberg operator, Eq. (2.8))

$$
\begin{aligned}
\text{tr}\{\exp(-\beta H_0)\rho(\mathbf{0}, 0)\rho(\mathbf{r}, t)\} &= \text{tr}\{\rho(\mathbf{r}, t)\exp(-\beta H_0)\rho(\mathbf{0}, 0)\} \\
&= \text{tr}\{\exp(-\beta H_0)\rho(\mathbf{r}, t - i\hbar\beta)\rho(\mathbf{0}, 0)\} \\
&= \exp(-i\hbar\beta\partial/\partial t)\text{tr}\{\exp(-\beta H_0)\rho(\mathbf{r}, t)\rho(\mathbf{0}, 0)\}.
\end{aligned}
\tag{3.6}
$$

In the classical limit, for β small, Eq. (3.3) becomes

$$\text{Im}\chi(\mathbf{k}, \omega) = (-n\beta\omega/2)S(\mathbf{k}, \omega). \tag{3.7}$$

IV. KRAMERS-KRONIG RELATIONS

Causality imposes that the response function $K(\mathbf{r}, t)$ vanish for $t < 0$. In other words the fluid is influenced only by the action of the external perturbation in the past. Introducing the "intermediate" response function $\chi(\mathbf{k}, t)$ as the space Fourier transform of $K(\mathbf{r}, t)$, we have

$$\chi(\mathbf{k}, t) = 0 \quad \text{for} \quad t < 0. \tag{4.1}$$

On the other hand

$$\chi(\mathbf{k}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp(-i\omega t)\chi(\mathbf{k}, \omega). \tag{4.2}$$

Extending the definition of $\chi(\mathbf{k}, \omega)$ from real to complex frequencies, we can calculate this integral through contour methods and for $t < 0$ we can close the contour with the semicircle at infinity above the real axis. The contribution from the integration on the semicircle vanishes since $\chi(\mathbf{k}, \omega) \propto \omega^{-2}$ at high frequency. So the causality property (4.1) is guaranteed if $\chi(\mathbf{k}, \omega)$ is analytic in the upper part of the complex frequency plane.

Let us now consider the integral

$$\oint \frac{\chi(\mathbf{k}, \omega')}{\omega - \omega'} d\omega' = 0, \tag{4.3}$$

on the contour Γ shown in Fig. 1. This contour integral vanishes due to the analiticity of $\chi(\mathbf{k}, \omega)$. The contribution from the semicircle at infinity is again zero, so that

$$P \int_{-\infty}^{\infty} d\omega' \frac{\chi(\mathbf{k}, \omega')}{\omega' - \omega} - i\pi\chi(\mathbf{k}, \omega) = 0, \tag{4.4}$$

where again P denotes the Cauchy principal part of the integral on the real frequency axis and the second term comes from the integration over the small semicircle around the point ω. If we now separate $\chi(\mathbf{k}, \omega)$ into its real and imaginary parts we find

$$P \int_{-\infty}^{\infty} d\omega' \frac{\text{Re}\chi(\mathbf{k}, \omega')}{\omega' - \omega} + \pi\text{Im}\chi(\mathbf{k}, \omega) = 0, \tag{4.5}$$

and

$$P \int_{-\infty}^{\infty} d\omega' \frac{\text{Im}\chi(\mathbf{k}, \omega')}{\omega' - \omega} - \pi\text{Re}\chi(\mathbf{k}, \omega) = 0. \tag{4.6}$$

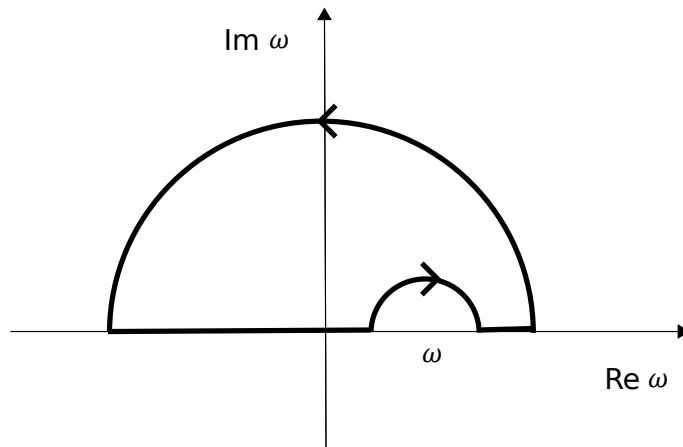These are the Kramers-Kronig relations.

FIG. 1. Integration contour on the complex ω plane.

V. THE DIELECTRIC FUNCTION

In a Coulomb liquid, the connection with the longitudinal dielectric function $\epsilon(k,\omega)$, becomes apparent from the Poisson equations

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = -4\pi e n_e(\mathbf{r},t), \tag{5.1}$$

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = -4\pi e[n_e(\mathbf{r},t) + \delta n(\mathbf{r},t)], \tag{5.2}$$

which yield

$$\frac{1}{\epsilon(k,\omega)} = \frac{\mathbf{k} \cdot \mathbf{E}(\mathbf{k},\omega)}{\mathbf{k} \cdot \mathbf{D}(\mathbf{k},\omega)} = 1 + \frac{\delta n(\mathbf{k},\omega)}{n_e(\mathbf{k},\omega)} = 1 + \frac{4\pi e^2}{k^2}\chi(k,\omega), \tag{5.3}$$

since from Eqs. (2.7) and (2.9) follows $\delta n(\mathbf{k},\omega) = \chi(k,\omega)V_b(\mathbf{k},\omega)$ where $\chi(k,\omega)$ is the Fourier transform of $K(|\mathbf{r} - \mathbf{r}'|, t - t')$ and

$$V_b(\mathbf{k},\omega) = \frac{4\pi e^2}{k^2}n_e(\mathbf{k},\omega). \tag{5.4}$$

Of course the field $\mathbf{E}$ and the associated screened or "Hartree" potential $V_H(\mathbf{k},\omega) = V_b(\mathbf{k},\omega)/\epsilon(k,\omega)$ would be experienced by a second test charge introduced into the plasma, rather than by the particles of the plasma. The latter also experience effects which involve the precise "hole" a particle of the plasma digs around itself. This latter effect brings about the so called local field corrections.

In addition to $\chi(\mathbf{k},\omega)$ which relates the displaced charge density to the potential *in vacuo*, it is useful to introduce yet another longitudinal response function, $\tilde{\chi}(k,\omega)$ say, by exploiting further the analogy with elementary electrostatics. This relates $n(\mathbf{k},\omega)$ directly to the Hartree potential through

$$n(\mathbf{k},\omega) = \tilde{\chi}(k,\omega)V_H(\mathbf{k},\omega). \tag{5.5}$$

We then have

$$\epsilon(k,\omega) = 1 - \frac{4\pi e^2}{k^2}\tilde{\chi}(k,\omega). \tag{5.6}$$

The expression $\chi(k,\omega) = \tilde{\chi}(k,\omega)/\epsilon(k,\omega)$ accounts at one stroke for the *long range* effects of the Coulomb interactions (the resonance at the plasma frequency, determined by $\epsilon(k,\omega) = 0$, is brought about explicitly in the denominator.

The simplest useful approximation to the dielectric function of the plasma is obtained by approximating $\tilde{\chi}$ by the density response function of an ideal gas. This corresponds to the Vlasov theory for the classical plasma and to the Lindhard theory for the degenerate electron fluid. Refinements of these theories aims at incorporating the effects of "exchange and correlation" in $\tilde{\chi}$. This expression being an abbreviation for the short range effects arising from the statistics ("exchange") and long range effect arising from the Coulomb interaction ("correlation"). Of course the exchange effects are absent in the classical limit.

VI. VERY DEGENERATE ATOMIC GASES

Consider a dilute (Bose) gas with interaction energy per unit area equal to U_0, confined to a 2D isotropic harmonic well with harmonic frequency ω_0. We will show that the kinetic and potential energy separately can be written in the general form $(T, V) = \hbar\omega_0 f_{(T,V)}(\alpha)$ where $\alpha = NU_0 m/\hbar^2$ is independent of ω_0. We will also show that the mean square radius of the cloud can be written in the form $d^2 N g(\alpha)$.

In the Hartree approximation, the ground state energy of the system is given by a Ginzburg-Pitaevskii-Gross energy functional

$$E_0[\psi(\mathbf{r})] = \int d^2\mathbf{r} \left[\frac{\hbar^2}{2m}|\nabla\psi|^2 + \frac{m\omega_0^2}{2}r^2|\psi|^2 + \frac{U_0}{2}|\psi|^4 \right], \tag{6.1}$$

where $\mathbf{r} = (x, y)$ and $\psi(\mathbf{r})$ is the order parameter.

The total ground state energy E_0 and wavefunction are determined minimizing $E_0[\psi]$ with the constraint

$$\int |\psi(\mathbf{r})| \, d^2\mathbf{r} = N, \tag{6.2}$$

where N is the number of particles in the gas.

Let's introduce the root mean square zero point displacement d

$$d = \sqrt{\frac{\hbar}{m\omega_0}}, \tag{6.3}$$

and let's make the following rescaling

$$\mathbf{r}_1 = \mathbf{r}/d, \tag{6.4}$$

$$\psi = f\frac{\sqrt{N}}{d}. \tag{6.5}$$

The energy functional becomes

$$E_0 = d^2 \int d^2\mathbf{r}_1 \left[\frac{\hbar^2}{2m}\left|\frac{\nabla_1 f}{d}\frac{\sqrt{N}}{d}\right|^2 + \frac{m\omega_0^2 d^2}{2}r_1^2\left|\frac{\sqrt{N}}{d}f\right|^2 + \frac{U_0}{2}\left|\frac{\sqrt{N}}{d}f\right|^4 \right] \tag{6.6}$$

$$= N\frac{\hbar\omega_0}{2} \int d^2\mathbf{r}_1[|\nabla_1 f|^2 + r_1^2|f|^2 + \alpha|f|^4], \tag{6.7}$$

where $\alpha = NU_0 m/\hbar^2$.

The order parameter normalization becomes,

$$N = \int d^2\mathbf{r}|\psi|^2 = d^2 \int d^2\mathbf{r}_1 \left|f\frac{\sqrt{N}}{d}\right|^2 = N \int d^2\mathbf{r}_1|f|^2, \tag{6.8}$$

or $\int d^2\mathbf{r}_1|f|^2 = 1$.

Now to find the total ground state energy we need to minimize

$$\delta\left\{ \int d^2\mathbf{r}_1[|\nabla_1 f|^2 + r_1^2|f|^2 + \alpha|f|^4] \right\} = 0, \tag{6.9}$$

with the constraint $\int d^2\mathbf{r}_1|f|^2 = 1$.

The function f_0 that solves the problem can only be a function of $\mathbf{r}_1$ and α, i.e. $f_0 = f_0(\mathbf{r}_1, \alpha)$.

So we can say

$$\frac{E_0}{N} = \hbar\omega_0\frac{1}{2} \int d^2\mathbf{r}_1[|\nabla_1 f_0|^2 + r_1^2|f_0|^2 + \alpha|f_0|^4] = \hbar\omega_0 f_E(\alpha), \tag{6.10}$$

$$\frac{V}{N} = \hbar\omega_0\frac{1}{2} \int d^2\mathbf{r}_1 r_1^2|f_0|^2 = \hbar\omega_0 f_V(\alpha), \tag{6.11}$$

$$\frac{U}{N} = \hbar\omega_0\frac{1}{2} \int d^2\mathbf{r}_1 \alpha|f_0|^4 = \hbar\omega_0 f_U(\alpha), \tag{6.12}$$

$$\frac{T}{N} = \hbar\omega_0\frac{1}{2} \int d^2\mathbf{r}_1 |\nabla_1 f_0|^2 = \hbar\omega_0 f_T(\alpha). \tag{6.13}$$

The mean square radius of the cloud is

$$\langle r^2 \rangle = \int d^2\mathbf{r}\, r^2 |\psi_0|^2 = d^2 N \int d^2\mathbf{r}_1\, r_1^2 |f_0|^2 = d^2 N g(\alpha). \tag{6.14}$$

We consider a "relaxing" simple harmonic oscillator, i.e. one in which the restoring force $F(t)$ is related to the displacement $x(t')$ for $t' < t$ by the formula

$$F(t) = \int K(t - t') x(t')\, dt', \tag{6.15}$$

with $K(t - t')$ given by

$$K(t - t') = \begin{cases} k_s \delta(t - t') - (k_s - k_t)\tau^{-1} \exp[-(t - t')/\tau] & t > t' \\ 0 & t < t' \end{cases}. \tag{6.16}$$

We find next the linear response function of the oscillator as a function of $\omega_t^2 = k_t/m$, $\omega_s^2 = k_s/m$, and τ, and find the damping of the oscillations in the limit $\omega_s \gtrsim \omega_t \gg \tau^{-1}$ and $\omega_s \gtrsim \omega_t \ll \tau^{-1}$.

Imagine to perturb the system with a force $G(t)$, the equation of motion of the harmonic oscillator will look like

$$m\ddot{x}(t) + F(t) = G(t). \tag{6.17}$$

The linear response function $\chi(\omega)$ is defined as

$$\chi(\omega) = \frac{\tilde{x}(\omega)}{\tilde{G}(\omega)}, \tag{6.18}$$

where we indicate with a tilde the time Fourier transform of the corresponding function. Taking the time Fourier transform of the equation of motion we get

$$\tilde{G} = -\omega^2 \tilde{x} + \omega_s^2 \tilde{x} - \frac{\omega_s^2 - \omega_t^2}{\tau} \tilde{x} \int_{-\infty}^{\infty} \theta(t) e^{-t/\tau} e^{i\omega t}\, dt, \tag{6.19}$$

where $\omega_s^2 = k_s/m$, $\omega_t^2 = k_t/m$, and we used the property of the Fourier transform to transform a convolution into a product.

Now we have

$$\int_{-\infty}^{\infty} \theta(t) e^{-t/\tau} e^{i\omega t}\, dt = \int_0^{\infty} e^{-t\left(\frac{1}{\tau} - i\omega\right)}\, dt = \frac{1}{\frac{1}{\tau} - i\omega}, \tag{6.20}$$

since $\tau > 0$.

So our equation of motion becomes

$$\tilde{G} = -\omega^2 \tilde{x} + \omega_s^2 \tilde{x} - \frac{\omega_s^2 - \omega_t^2}{1 - i\omega\tau} \tilde{x}, \tag{6.21}$$

and the linear response function looks like

$$\chi(\omega) = \frac{\tilde{x}(\omega)}{\tilde{G}(\omega)} = \frac{1}{-\omega^2 + \omega_s^2 - \frac{\omega_s^2 - \omega_t^2}{1 - i\omega\tau}} \tag{6.22}$$

Introducing adimensional frequencies $\bar{\omega} = \omega\tau$, $\bar{\omega}_{s,t} = \omega_{s,t}\tau$ we get

$$\chi(\bar{\omega}) = -\tau^2 \left(\frac{\bar{\omega} + i}{\bar{\omega}^3 + i\bar{\omega}^2 - \bar{\omega}\bar{\omega}_s^2 - i\bar{\omega}_t^2} \right). \tag{6.23}$$

To find the damping of the oscillations we need to find the poles of $\chi(\bar{\omega})$.

Note that

$$x(t) = \int_{-\infty}^{\infty} \chi(t - t') G(t')\, dt' \tag{6.24}$$

this gives us two informations about the poles of $\chi(\omega)$ extended from real to complex frequencies:

i. Causality implies that $\chi(t) = 0$ for $t < 0$, i.e.

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}\chi(\omega) = 0 \quad \text{for } t < 0, \tag{6.25}$$

which means that $\chi(\omega)$ cannot have poles in the upper half of the complex frequency plane (has to be analytic there).

ii. Since $\chi(t)$ and $G(t)$ are physical quantities, hence real, $\chi(-\omega^*) = \chi^*(\omega)$.

Now, since the denominator of $\chi(\omega)$ is a cubic, it has to have 3 roots. Because of i. and ii. these 3 roots have to be of the following kind

$$\bar{\omega}_1 = -ia,$$
$$\bar{\omega}_2 = -\bar{\omega}_3^* = -ic + b,$$

with a, b, c three real positive numbers. We want to find the damping of the oscillations produced by the term b in $\chi(t)$. That is, we want to find c (the $\omega = \bar{\omega}_1/\tau$ pole will give rise to a dumped non oscillating factor instead).

So we want

$$\bar{\omega}^3 + i\bar{\omega}^2 - \bar{\omega}\bar{\omega}_s^2 - i\bar{\omega}_t^2$$
$$= (\bar{\omega} + ia)(\bar{\omega} + ic + b)(\bar{\omega} + ic - b)$$
$$= (\bar{\omega} + ia)(\bar{\omega}^2 - c^2 + 2ic\bar{\omega} - b^2)$$
$$= \bar{\omega}^3 + i\bar{\omega}^2(2c + a) - \bar{\omega}(b^2 + c^2 + 2ca) - i(b^2 + c^2)a,$$

which gives

$$\begin{cases} 2c + a = 1 \\ (b^2 + c^2) + 2ca = \bar{\omega}_s^2 \\ (b^2 + c^2)a = \bar{\omega}_t^2 \end{cases} \tag{6.26}$$

This linear system reduces to

$$c^3 - c^2 + c\left(\frac{1 + \bar{\omega}_s^2}{4}\right) - \left(\frac{\bar{\omega}_s^2 - \bar{\omega}_t^2}{8}\right) = 0 \tag{6.27}$$

Now for small τ Eq. (6.27) reduces to

$$c^3 - c^2 + c/4 \approx 0, \tag{6.28}$$

which has solutions $c = 0, a = 1, b = \bar{\omega}_t$ and $c = 1/2, a = 0, b = \sqrt{\bar{\omega}_s^2 - 1/4}$. Both these solutions are uninteresting since the first one gives

$$\chi(\bar{\omega}) \approx -\tau^2 \frac{1}{(\bar{\omega} + \bar{\omega}_t)(\bar{\omega} - \bar{\omega}_t)}, \tag{6.29}$$

which corresponds to an undamped

$$\chi(t) \approx (\tau/2\omega_t)\sin(\omega_t t). \tag{6.30}$$

While the second one gives a complex b, since $\bar{\omega}_s^2 \ll 1/4$, which cannot be.
We then learn that for the case $\bar{\omega}_s^2 \gtrsim \bar{\omega}_t^2 \ll 1$ we need to expand around $c = 0$ to obtain

$$c\left(\frac{1}{4}\right) - \left(\frac{\bar{\omega}_s^2 - \bar{\omega}_t^2}{8}\right) \approx 0, \tag{6.31}$$

with solution

$$c \approx \frac{\bar{\omega}_s^2 - \bar{\omega}_t^2}{2}, \tag{6.32}$$

and $a \approx 1$ and $b \approx \bar{\omega}_s$. So we have

$$\chi(\bar{\omega}) \approx -\tau^2 \frac{1}{\left[\bar{\omega} + i\left(\frac{\bar{\omega}_s^2 - \bar{\omega}_t^2}{2}\right) + \bar{\omega}_s\right]\left[\bar{\omega} + i\left(\frac{\bar{\omega}_s^2 - \bar{\omega}_t^2}{2}\right) - \bar{\omega}_s\right]}, \tag{6.33}$$

which corresponds to

$$\chi(t) \approx \tau e^{-\left(\frac{\omega_s^2 - \omega_t^2}{2}\right)\tau t} \frac{\sin(\omega_s t)}{\omega_s}. \tag{6.34}$$

For large τ Eq. (6.27) reduces to

$$c\left(\frac{\bar{\omega}_s^2}{4}\right) - \left(\frac{\bar{\omega}_s^2 - \bar{\omega}_t^2}{8}\right) \approx 0, \tag{6.35}$$

which gives

$$c \approx \frac{1}{2}\left(\frac{\bar{\omega}_s^2 - \bar{\omega}_t^2}{\bar{\omega}_s^2}\right). \tag{6.36}$$

Now for $\bar{\omega}_s^2 \gtrsim \bar{\omega}_t^2 \gg 1$ c is small and from Eq. (6.26) we find $a \approx 1$ and $b \approx \bar{\omega}_s$. So we have

$$\chi(\bar{\omega}) \approx -\tau^2 \frac{1}{\left[\bar{\omega} + i\left(\frac{\bar{\omega}_s^2 - \bar{\omega}_t^2}{2\bar{\omega}_s^2}\right) + \bar{\omega}_s\right]\left[\bar{\omega} + i\left(\frac{\bar{\omega}_s^2 - \bar{\omega}_t^2}{2\bar{\omega}_s^2}\right) - \bar{\omega}_s\right]}, \tag{6.37}$$

which corresponds to

$$\chi(t) \approx \tau e^{-\left(\frac{\omega_s^2 - \omega_t^2}{2\omega_s^2}\right)t/\tau} \frac{\sin(\omega_s t)}{\omega_s}. \tag{6.38}$$

Consider a (boson) system living in a plane with Hamiltonian

$$\hat{H}_0 = \sum_{i=1}^{N}\left[\frac{\hat{\mathbf{P}}_i^2}{2m} + \hat{V}_{ext}(\hat{\mathbf{r}}_i)\right] + \frac{U_0}{2}\sum_{i,j=1}^{N}\delta^2(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j), \tag{6.39}$$

$$\hat{V}_{ext} = \frac{1}{2}m\omega_0^2\sum_{i=1}^{N}\hat{\mathbf{r}}_i^2. \tag{6.40}$$

We can show that for this system the frequency of the monopole mode is exactly $2\omega_0$ irrespective of the value of $\alpha = NU_0 m/\hbar^2$.

Since we are in a plane $\hat{\mathbf{r}}_i = (\hat{r}_{i_1}, \hat{r}_{i_2})$ and $\hat{\mathbf{p}}_i = (\hat{p}_{i_1}, \hat{p}_{i_2})$. It is useful to introduce the following two operators

$$\hat{Q}_\alpha = \sum_{i=1}^{N}\hat{r}_{i_\alpha}^2, \tag{6.41}$$

$$\hat{\Lambda}_\alpha = \frac{1}{2m}\sum_{i=1}^{N}[\hat{p}_{i_\alpha}\hat{r}_{i_\alpha} + \hat{r}_{i_\alpha}\hat{p}_{i_\alpha}], \tag{6.42}$$

for each dimension $\alpha = 1, 2$ and the usual density and current operators

$$\hat{\rho}(\mathbf{r}) = \sum_{i=1}^{N}\delta(\mathbf{r} - \hat{\mathbf{r}}_i), \tag{6.43}$$

$$\hat{\mathbf{J}}(\mathbf{r}) = \frac{1}{2m}\sum_{i=1}^{N}[\hat{\mathbf{p}}_i\delta(\mathbf{r} - \hat{\mathbf{r}}_i) + \delta(\mathbf{r} - \hat{\mathbf{r}}_i)\hat{\mathbf{p}}_i], \tag{6.44}$$

So that for example

$$\hat{V}_{ext} = \frac{1}{2} m \omega_0^2 \sum_{\alpha=1}^{2} \hat{Q}_\alpha. \tag{6.45}$$

This *harmonic trap* may be perturbed by a harmonic perturbation $\hat{H}' = \lambda \sum_{\alpha=1}^{2} \hat{Q}_\alpha$. This we will do next.

A. Moments Sum Rules

Imagine we apply to the system $\hat{H}_0$ a perturbation

$$\hat{H}'(t) = \sum_{\alpha=1}^{2} \int d^2 \mathbf{r} \, \hat{A}_\alpha(\mathbf{r}) \lambda_\alpha(\mathbf{r}, t), \tag{6.46}$$

where $\lambda_\alpha(\mathbf{r}, t)$ are external fields and $\hat{A}_\alpha(\mathbf{r})$ are observables of the system coupled to the fields. Our system now is $\hat{H} = \hat{H}_0 + \hat{H}'$. We can then apply the formalism developed in Sec. II for the linear response theory.

Let us consider the moments of the dissipation spectrum, defined as

$$M_{\alpha\beta}^n(k) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \omega^n \, \text{Im}\chi_{\alpha\beta}(k, \omega). \tag{6.47}$$

Since $\text{Im}\chi_{\alpha\beta}(k, \omega)$ is an odd function of ω due to Eq. (3.3), the even moments vanish. We will just be interested in the first two non-zero moments $n = 1, 3$. The moments are related to the fluctuations of the observables $\hat{A}_\alpha(\mathbf{r})$ by the fluctuation-dissipation theorem in the following way

$$M_{\alpha\beta}^n(k) = -\frac{2}{(-i)^n} \left. \frac{\partial^n \chi_{\alpha\beta}''(k, t)}{\partial t^n} \right|_{t=0}, \tag{6.48}$$

$$\chi_{\alpha\beta}''(k, t) = -\frac{1}{2\hbar} \langle \left[\hat{A}_\alpha(\mathbf{k}, t), \hat{A}_\beta(-\mathbf{k}, 0) \right] \rangle_0, \tag{6.49}$$

where we used Eq. (2.13), $\langle \ldots \rangle_0$ denotes the expectation value over the ground state of the unperturbed system $\hat{H}_0$, $[\ldots]$ is the commutator, $\hat{A}_\alpha(\mathbf{k}, t)$ is the Heisenberg representation of the spatial Fourier transform of $\hat{A}_\alpha(\mathbf{r})$ such that

$$\frac{\partial \hat{A}(t)}{\partial t} = [\hat{A}(t), \hat{H}_0]/i\hbar. \tag{6.50}$$

Let us write down the first two non-zero ones

$$M_{\alpha\beta}^1(k) = \frac{1}{\hbar^2} \langle \left[\left[\hat{A}_\alpha, \hat{H}_0 \right], \hat{A}_\beta \right] \rangle_0, \tag{6.51}$$

$$M_{\alpha\beta}^3(k) = \frac{1}{\hbar^4} \langle \left[\left[\left[\hat{A}_\alpha, \hat{H}_0 \right], \hat{H}_0 \right], \hat{H}_0 \right], \hat{A}_\beta \right] \rangle_0, \tag{6.52}$$

where all the $\hat{A}$ observables are taken at $(k, 0)$.

Let us now choose, in Eq. (6.46), the external field independent of time, space, and index α, and

$$\lambda_\alpha(\mathbf{r}, t) = \lambda = \text{constant}, \tag{6.53}$$
$$\hat{A}_\alpha(\mathbf{r}) = r_\alpha^2 \hat{\rho}(\mathbf{r}), \tag{6.54}$$

with $\hat{\rho}$ defined in Eq. (6.43). We can then evaluate the first and third moments at $k = 0$. Let us start with $M_{\alpha\beta}^1(k = 0)$. Proceeding step by step, we find

$$\left[\hat{Q}_\alpha, \hat{H}_0 \right] = 2i\hbar \hat{\Lambda}_\alpha, \tag{6.55}$$

$$\left[\hat{Q}_\beta, \left[\hat{Q}_\alpha, \hat{H}_0 \right] \right] = \left[\hat{Q}_\beta, 2i\hbar \hat{\Lambda}_\alpha \right] = -\frac{4\hbar^2}{m} \delta_{\alpha\beta} \hat{Q}_\alpha, \tag{6.56}$$

where $\hat{Q}_\alpha$ and $\hat{\Lambda}_\alpha$ are defined in Eqs. (6.41) and (6.42) and we used the usual commutation relations $[\hat{r}_{i_\alpha}, \hat{r}_{j_\beta}] = [\hat{p}_{i_\alpha}, \hat{p}_{j_\beta}] = 0$, $[\hat{r}_{i_\alpha}, \hat{p}_{j_\beta}] = i\hbar\delta_{ij}\delta_{\alpha\beta}$, so that only the kinetic energy term in $\hat{H}_0$ contributes to the first moment.

So using Eq. (6.56) into Eq. (6.51) we find at $k = 0$

$$M^1_{\alpha\beta}(0) = \frac{4}{m}\delta_{\alpha\beta}\langle\hat{Q}_\alpha\rangle_0. \tag{6.57}$$

In order to find the third moment is convenient to rearrange the commutators in Eq. (6.52) as follows

$$M^3_{\alpha\beta}(k) = \frac{1}{\hbar^4}\langle\left[\left[\hat{A}_\beta, \hat{H}_0\right], \left[\left[\hat{A}_\alpha, \hat{H}_0\right], \hat{H}_0\right]\right]\rangle_0, \tag{6.58}$$

so that at $k = 0$ one finds from Eq. (6.55)

$$M^3_{\alpha\beta}(0) = -\frac{4}{\hbar^2}\langle\left[\hat{\Lambda}_\beta, \left[\hat{\Lambda}_\alpha, \hat{H}_0\right]\right]\rangle_0. \tag{6.59}$$

Let us start by calculating the commutator $\left[\hat{\Lambda}_\alpha, \hat{U}\right]$ where

$$\hat{U} = \frac{U_0}{2}\int d^2\mathbf{r}\,\hat{\rho}^2(\mathbf{r}) = \frac{U_0}{2}\sum_{i,j=1}^{N}\delta^2(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j), \tag{6.60}$$

$$\hat{\Lambda}_\alpha = \int d^2\mathbf{r}\,r_\alpha\hat{J}_\alpha(\mathbf{r}). \tag{6.61}$$

From Eq. (A1) in Appendix we get

$$\left[\hat{\Lambda}_\alpha, \hat{U}\right] = \frac{i\hbar}{m}U_0\int d^2\mathbf{r}\,\hat{\rho}(\mathbf{r})\nabla_\alpha\left[r_\alpha\hat{\rho}(\mathbf{r})\right]. \tag{6.62}$$

Then, from the identity

$$\hat{\rho}(\mathbf{r})\nabla_\alpha\left[r_\alpha\hat{\rho}(\mathbf{r})\right] = \frac{1}{2}\{\hat{\rho}^2(\mathbf{r}) + \nabla_\alpha\left[r_\alpha\hat{\rho}^2(\mathbf{r})\right]\}, \tag{6.63}$$

and from the boundedness of the system follows

$$\left[\hat{\Lambda}_\alpha, \hat{U}\right] = \frac{i\hbar}{m}\frac{U_0}{2}\int d^2\mathbf{r}\,\hat{\rho}^2(\mathbf{r}) = \frac{i\hbar}{m}\hat{U}, \tag{6.64}$$

where in the last equality we used Eq. (6.60).

We have already calculated the commutator with the external potential

$$\begin{aligned}\left[\hat{\Lambda}_\alpha, \hat{V}_{ext}\right] &= \left[\hat{\Lambda}_\alpha, \tfrac{1}{2}m\omega_0^2\sum_\beta\hat{Q}_\beta\right] \\ &= -\frac{2i\hbar}{m}\left(\tfrac{1}{2}m\omega_0^2\sum_\beta\hat{Q}_\beta\delta_{\alpha\beta}\right) \\ &= -\frac{2i\hbar}{m}\hat{V}_\alpha \end{aligned} \tag{6.65}$$

where in the second equality we used Eq. (6.56) and the last equality defines $\hat{V}_\alpha = \frac{1}{2}m\omega_0^2\sum_i\hat{r}_{i_\alpha}^2$.

All that is left is to calculate the commutator $\left[\hat{\Lambda}_\alpha, \hat{K}\right]$ where $\hat{K} = \sum_i\hat{\mathbf{p}}_i^2/2m$ is the kinetic energy operator. So now we get

$$\left[\hat{\Lambda}_\alpha, \hat{K}\right] = \frac{i\hbar}{m^2}\sum_i\hat{p}_{i_\alpha}^2 = \frac{2i\hbar}{m}\sum_i\hat{K}_\alpha, \tag{6.66}$$

where in the last equality we have defined $\hat{K}_\alpha = \sum_i\hat{p}_{i_\alpha}^2/2m$.

So, collecting all the pieces

$$\left[\hat{\Lambda}_\alpha, \hat{H}_0\right] = \frac{i\hbar}{m}\left\{2\hat{K}_\alpha - 2\hat{V}_\alpha + \hat{U}\right\}. \tag{6.67}$$

Now taking the ground state expectation value of this commutator one gets the virial theorem since $\langle\left[\hat{\Lambda}_\alpha, \hat{H}_0\right]\rangle_0 = \langle\hat{\Lambda}_\alpha\rangle_0 E - E\langle\hat{\Lambda}_\alpha\rangle_0 = 0$, namely

$$2\langle\hat{K}_\alpha\rangle_0 - 2\langle\hat{V}_\alpha\rangle_0 + \langle\hat{U}\rangle_0 = 0. \tag{6.68}$$

We can now determine the last necessary commutator (using Eq. (6.67))

$$\left[\hat{\Lambda}_\beta, \left[\hat{\Lambda}_\alpha, \hat{H}_0\right]\right] = \frac{i\hbar}{m}\left[\hat{\Lambda}_\beta, 2\hat{K}_\alpha - 2\hat{V}_\alpha + \hat{U}\right] \tag{6.69}$$

$$= \left(\frac{i\hbar}{m}\right)^2\left\{4\hat{K}_\alpha\delta_{\alpha\beta} + 4\hat{V}_\alpha\delta_{\alpha\beta} + \hat{U}\right\}. \tag{6.70}$$

So that in the end, using this result into Eq. (6.59), we find for the third moment

$$M^3_{\alpha\beta}(0) = \frac{8}{m}\left\langle\delta_{\alpha\beta}\left[2\hat{K}_\alpha + 2\hat{V}_\alpha\right] + \frac{1}{2}\hat{U}\right\rangle_0. \tag{6.71}$$

Now, a *collective excitation* at frequency ω_α manifests itself with the appearance of a $\delta(\omega - \omega_\alpha)$ in the dissipation spectrum, i.e. $\text{Im}\chi_{\alpha\beta}(0,\omega) \propto \delta(\omega - \omega_\alpha)\delta_{\alpha\beta}$.

From the definition of the moments, Eq. (6.47) follows

$$M^1_{\alpha\beta} \propto \omega_\alpha\delta_{\alpha\beta}, \tag{6.72}$$

$$M^3_{\alpha\beta} \propto \omega^3_\alpha\delta_{\alpha\beta}, \tag{6.73}$$

from which follows

$$(`M^3/M^1`)_{\alpha\beta} = \omega^2_\alpha\delta_{\alpha\beta}. \tag{6.74}$$

So we need to take the "ratio" 'M^3/M^1' and eventually diagonalize the matrix. From Eqs. (6.10)-(6.13) and using Feynman theorem we find

$$\frac{\partial\langle\hat{H}_0\rangle_0}{\partial\omega^2_0}\bigg|_m = \left\langle\frac{\partial\hat{H}_0}{\partial\omega^2_0}\bigg|_m\right\rangle_0 = \frac{\langle\hat{V}_{ext}\rangle_0}{\omega^2_0}, \tag{6.75}$$

from which follows $f_E = 2f_V$. This is nothing else than the virial theorem again $2f_V = f_K + f_V + f_U$ or $f_K - f_V + f_U = 0$ (which should be compared with Eq. (6.68) which holds for $\alpha = 1, 2$). Here we use the same notation used in Eqs. (6.10)-(6.13) with the new interpretation that $f_E = \langle\hat{H}_0\rangle_0, f_K = \langle\hat{K}\rangle_0, f_V = \langle\hat{V}_{ext}\rangle_0, f_U = \langle\hat{U}\rangle_0$. Using Feynman theorem again, we also find

$$\frac{\partial\langle\hat{H}_0\rangle_0}{\partial m}\bigg|_{\omega_0} = \left\langle\frac{\partial\hat{H}_0}{\partial m}\bigg|_{\omega_0}\right\rangle_0 = \frac{\langle\hat{V}_{ext}\rangle_0 - \langle\hat{K}\rangle_0}{m}, \tag{6.76}$$

from which follows $\partial f_E/\partial m = (f_V - f_K)/m$.

Using Eqs. (6.75) and (6.76) together

$$1 - \frac{f_K}{f_V} = 2\left(m\frac{\partial f_E}{\partial m}\frac{1}{f_E}\right) = 2\delta, \tag{6.77}$$

which defines δ.

Let us now write M^3/M^1 using f_K/f_V. from Eqs. (6.57) and (6.71) follows

$$M^1_{\alpha\beta}(0) = \delta_{\alpha\beta}\frac{4}{m\omega^2_0}\langle\hat{V}_{ext}\rangle_0, \tag{6.78}$$

$$M^3_{\alpha\beta}(0) = \frac{8}{m}\left\{\delta_{\alpha\beta}\left(2\langle\hat{K}_\alpha\rangle_0 + 2\langle\hat{V}_\alpha\rangle_0\right) + \frac{1}{2}\langle\hat{U}\rangle_0\right\}$$

$$= \frac{8}{m}\left\{\delta_{\alpha\beta}\left(\langle\hat{K}\rangle_0 + \langle\hat{V}_{ext}\rangle_0\right) + \frac{1}{2}\langle\hat{U}\rangle_0\right\}$$

$$= \frac{4}{m}\left\{\delta_{\alpha\beta}2(\langle\hat{K}\rangle_0 + \langle\hat{V}_{ext}\rangle_0) + (\langle\hat{V}_{ext}\rangle_0 - \langle\hat{K}\rangle_0)\right\}, \tag{6.79}$$

where for the third moment we used isotropy in the first equality and the virial theorem in the second. Upon rescaling M^3

$$
\begin{aligned}
('M^3/M^1')_{\alpha\beta} &= \omega_0^2 \left\{ \delta_{\alpha\beta} 2 \left(\frac{f_K}{f_V} + 1 \right) + \left(1 - \frac{f_K}{f_V} \right) \right\} \\
&= \omega_0^2 \left\{ \delta_{\alpha\beta} 2(2 - 2\delta) + 2\delta \right\} \\
&= \omega_0^2 \begin{pmatrix} 4 - 2\delta & 2\delta \\ 2\delta & 4 - 2\delta \end{pmatrix}.
\end{aligned} \tag{6.80}
$$

We can now diagonalize this matrix. The eigenvalues are

$$
\begin{aligned}
4 - 2\delta - \lambda_\pm &= \pm 2\delta, \\
\lambda_\pm &= 4 - 2\delta \mp 2\delta,
\end{aligned} \tag{6.81}
$$

and the eigenvectors $\mathbf{v}_\pm = (1, \pm 1)$. So that the monopole mode collective excitation has exactly a frequency

$$
\omega_+ = \sqrt{\lambda_+ \omega_0^2} = \boxed{2\omega_0}. \tag{6.82}
$$

VII. CONCLUSIONS

In conclusion we reviewed the linear response theory for quantum liquids with the associated Kramers-Kronig relations due to causality, we defined the longitudinal dielectric function for Coulomb liquids, and we determined the monopole frequency for an atomic gas in a plane in a harmonic trap. The result we obtained through linear response theory is exact because we let the perturbation vanish in the end.

Appendix A: A particular commutator

Given any two functions $f(\mathbf{r})$ and $g(\mathbf{r})$, we have

$$
\begin{aligned}
&[\int d^2\mathbf{r}\, \hat{J}_\alpha(\mathbf{r}) g(\mathbf{r}), \int d^2\mathbf{r}'\, \hat{\rho}(\mathbf{r}') f(\mathbf{r}')] = \\
&-\frac{i\hbar}{m} \int d^2\mathbf{r}\, g(\mathbf{r}) \hat{\rho}(\mathbf{r}) \nabla f(\mathbf{r}) = \\
&\frac{i\hbar}{m} \int d^2\mathbf{r}\, f(\mathbf{r}) \nabla \left[g(\mathbf{r}) \hat{\rho}(\mathbf{r}) \right],
\end{aligned} \tag{A1}
$$

where the operators $\hat{\rho}$ and $\hat{J}_\alpha$ are defined in Eqs. (6.43) and (6.44) of the main text and in the last equality we have used the fact that our fluid is bounded (i.e. the density operator decays to zero at $r \to \infty$) so that in the integration by parts we can neglect the surface term.

AUTHOR DECLARATIONS

Conflict of interest

The author has no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

---

[1] D. P. e Ph. Nozières, *The Teory of Quantum Liquids* (W. A. Benjamin, Inc., New York, 1966) chapter 2.

[2] N. H. March and M. P. Tosi, *Coulomb Liquids* (Academic Press, London, 1984) appendix 5.

[3] J. P. Hansen and I. R. McDonald, *Theory of Simple Liquids*, 2nd ed. (Academic Press, London, 1986) section 7.6.