Statistical Gravity, ADM splitting, and AQ

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I propose a possible way to render numerically accessible the path integral Monte Carlo computations required in the Statistical Gravity theory described in a recent publication [Riccardo Fantoni, Quantum Reports, **6**, 706 (2024)]. This requires the use of the Arnowitt, Deser, and Misner (ADM) splitting and of the Affine Quantization (AQ) method.

Keywords: General Relativity; Einstein-Hilbert action; ADM splitting; Affine Quantization; Statistical Physics; Path Integral; Monte Carlo

I. INTRODUCTION

The idea to realize a quantum theory of gravity has a long history [1, 2]. Recently we proposed a theory for *Statistical Gravity* [3], the FEBB. Leaving aside the feasible experimental confirmations for it, it is yet important to prove that it gives rise to quantities (observables thermal averages) that are mathematically well defined and can therefore be computed (at least numerically). We are thinking, for example, at the problems that one may encounter in computing a *constrained* quantum field theory [4–16], even the simplest one as the scalar (relativistic euclidean). In these cases we could experience how important it was to use the method of *Affine Quantization* (AQ) (as opposed to the canonical quantization) in order to render the particular theory *non trivial*. But even before worrying about the renormalizability of the particular quantum field theory it makes sense to worry about the soundness of the place it occupies in the underlying Hilbert space.

With this in mind, in this short paper, following the idea already put forward in Ref. [12] for a construction of a well defined Quantum Gravity, we propose to use the method of AQ also to construct a well defined Statistical Gravity.

In these complex tensorial quantum field theories, even the determination of the relevant *semiclassical action* can become a formidable task due to the intertwining of the tensorial calculus and the commutation calculus. Here we will not carry out any of this necessary complex calculus explicitly but will just lay down the problem showing that it is a well defined one.

II. EINSTEIN'S FIELD EQUATIONS FROM A VARIATIONAL PRINCIPLE

Sempre caro mi fu quest'ermo colle,
e questa siepe, che da tanta parte
dell'ultimo orizzonte il guardo esclude.

Giacomo Leopardi
L' Infinito

The Einstein-Hilbert action in general relativity is the action that yields the Einstein field equations through the stationary-action principle. With the $(-+++)$ metric signature, the gravitational part of the action is given as [17, 18]

$$S = \frac{1}{2\kappa}\int R\sqrt{-g}\,\mathrm{d}^4x, \tag{2.1}$$

where $g \equiv \det(g_{\mu\nu})$ is the determinant of the metric tensor matrix, $\sqrt{-g}$ is the scalar density, $x \equiv (ct, \mathbf{x})$ is an event with $t \equiv x^0/c$ time and $\mathbf{x} \equiv (x^1, x^2, x^3)$ a point in space, $\sqrt{-g}\,\mathrm{d}^4x$ is the invariant "volume" element, R is the Ricci scalar, and $\kappa = 8\pi Gc^{-4}$ is the Einstein gravitational constant (G is the gravitational constant and c is the speed of light in vacuum). If it converges, the integral is taken over the whole spacetime. If it does not converge, S is no longer

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well-defined, but a modified definition where one integrates over arbitrarily large, relatively compact domains, still yields the Einstein equation as the Euler-Lagrange equation of the Einstein-Hilbert action. The action was proposed [17] by David Hilbert in 1915 as part of his application of the variational principle to a combination of gravity and electromagnetism.

The stationary-action principle then tells us that to recover a physical law, we must demand that the variation of this action with respect to the inverse metric be zero, yielding

$$
\begin{aligned}
0 = \delta S \; &= \int \left[\frac{1}{2\kappa} \frac{\delta\left(\sqrt{-g}R\right)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \, \mathrm{d}^4x && (2.2)\\
&= \int \left[\frac{1}{2\kappa} \left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) \right] \delta g^{\mu\nu} \sqrt{-g} \, \mathrm{d}^4x. && (2.3)
\end{aligned}
$$

Since this equation should hold for any variation $\delta g^{\mu\nu}$, it implies that

$$
\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = 0 \tag{2.4}
$$

is the equation of motion for the metric field.

The variation of the Ricci scalar in Eq. (2.4) follows from varying the Riemann curvature tensor, and then the Ricci curvature tensor. The first step is captured by the Palatini identity

$$
\delta R_{\sigma\nu} \equiv \delta R^{\rho}{}_{\sigma\rho\nu} = \left(\delta\Gamma^{\rho}_{\nu\sigma}\right)_{;\rho} - \left(\delta\Gamma^{\rho}_{\rho\sigma}\right)_{;\nu} . \tag{2.5}
$$

Using the product rule, the variation of the Ricci scalar $R = g^{\sigma\nu} R_{\sigma\nu}$ then becomes,

$$
\begin{aligned}
\delta R &= R_{\sigma\nu}\delta g^{\sigma\nu} + g^{\sigma\nu}\delta R_{\sigma\nu} \\
&= R_{\sigma\nu}\delta g^{\sigma\nu} + \left(g^{\sigma\nu}\delta\Gamma^{\rho}_{\nu\sigma} - g^{\sigma\rho}\delta\Gamma^{\mu}_{\mu\sigma}\right)_{;\rho} ,
\end{aligned} \tag{2.6}
$$

where we also used the metric compatibility $g^{\mu\nu}_{;\sigma} = 0$, and renamed the summation indices $(\rho,\nu) \to (\mu,\rho)$ in the last term. When multiplied by $\sqrt{-g}$, the term $\left(g^{\sigma\nu}\delta\Gamma^{\rho}_{\nu\sigma} - g^{\sigma\rho}\delta\Gamma^{\mu}_{\mu\sigma}\right)_{;\rho}$ becomes a total derivative, since for any vector A^{λ} and any tensor density $\sqrt{-g}\,A^{\lambda}$, we have

$$
\sqrt{-g}\,A^{\lambda}_{;\lambda} = \left(\sqrt{-g}\,A^{\lambda}\right)_{;\lambda} = \left(\sqrt{-g}\,A^{\lambda}\right)_{,\lambda} . \tag{2.7}
$$

By Stokes' theorem, this only yields a boundary term when integrated. The boundary term is in general non-zero, because the integrand depends not only on $\delta g^{\mu\nu}$, but also on its partial derivatives $\partial_{\lambda}\,\delta g^{\mu\nu} \equiv \delta\,\partial_{\lambda} g^{\mu\nu}$. However, when the variation of the metric $\delta g^{\mu\nu}$ vanishes in a neighbourhood of the boundary or when there is no boundary, this term does not contribute to the variation of the action. Thus, we can forget about this term and simply obtain

$$
\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}. \tag{2.8}
$$

at events not in the closure of the boundary.

The variation of the determinant in Eq. (2.4) requires Jacobi's formula, the rule for differentiating a determinant:

$$
\delta g = g g^{\mu\nu}\delta g_{\mu\nu}. \tag{2.9}
$$

Using this we get

$$
\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\delta g_{\mu\nu}\right) = -\frac{1}{2}\sqrt{-g}\left(g_{\mu\nu}\delta g^{\mu\nu}\right) \tag{2.10}
$$

In the last equality we used the fact that from the symmetry of the metric tensor and $g_{\mu\nu}g^{\nu\mu} = \delta^{\mu}_{\mu} = 4$ follows

$$
g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu} \tag{2.11}
$$

Thus we conclude that

$$
\frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu}. \tag{2.12}
$$

Now that we have all the necessary variations at our disposal, we can insert Eq. (2.12) and Eq. (2.8) into the equation of motion (2.4) for the metric field to obtain

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{2.13}$$

which is the Einstein field equations in vacuum.

Moreover, since Einstein's tensor $G_{\mu\nu}$ appears from a variational principle:

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{2\kappa}\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = \frac{1}{2\kappa} G_{\mu\nu}, \tag{2.14}$$

its covariant divergence is necessarily zero [18].

Matter or electromagnetic fields will produce a curvature of spacetime. In order to take this into account it is necessary to add a term $\mathcal{L}_F$ as follows,

$$S = \int \left(\frac{1}{2\kappa} R + \mathcal{L}_{\mathrm{F}}\right) \sqrt{-g}\, \mathrm{d}^4 x. \tag{2.15}$$

The equations of motion coming from the stationary-action principle now become

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{2.16}$$

where

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{F}})}{\delta g^{\mu\nu}} = -2\frac{\delta \mathcal{L}_{\mathrm{F}}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\mathrm{F}}, \tag{2.17}$$

is the stress-energy tensor and $\kappa = 8\pi G/c^4$ has been chosen such that the non-relativistic limit yields the usual form of Newton's gravity law.

III. ADM 3+1 FOLIATION OF SPACETIME

Ma sedendo e mirando, interminati
spazi di là da quella, e sovrumani
silenzi, e profondissima quiete
io nel pensier mi fingo, ove per poco
il cor non si spaura.

Giacomo Leopardi
L' Infinito

Arnowitt, Deser and Misner (ADM) proposed in 1962 the following 3+1 foliation of spacetime [19]

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{3.1}$$

where now Latin indexes run over the three spatial components $1, 2, 3$. They called N the *lapse* and N_i the *shift*. To split the time component from the 3 spatial components they chose the following

$$||g_{\mu\nu}|| = \begin{pmatrix} -(N^2 - N^i N_i) & N_i \\ N_i & g_{ij} \end{pmatrix}, \tag{3.2}$$

$$||g^{\mu\nu}|| = \begin{pmatrix} -1/N^2 & N^i/N^2 \\ N^i/N^2 & g^{ij} - N^i N^j/N^2 \end{pmatrix}, \tag{3.3}$$

which are inverse by sight. Note also that $\sqrt{-{}^4g} = N\sqrt{{}^3g}$ where ${}^3g = \det\{g_{ij}\}$ and ${}^4g = \det\{g_{\mu\nu}\}$ and we indicate with a presuperscript 4 the full four dimensional tensor and with a presuperscript 3 the spatial 3×3 tensor, when strictly necessary to avoid confusion. Therefore we will raise (or lower) Greek indexes with the full metric tensor $g^{\mu\nu}$ and Latin indexes with the spatial metric tensor g^{ij} which also satisfies $g_{ik}g^{kj} = \delta_i^j$.

ADM showed that if one chooses as generalized coordinate g_{ij} and conjugated momentum

$$\pi^{ij} \equiv \sqrt{-{}^4g}(\Gamma_p{}^0{}_q - g_{pq}\Gamma_r{}^0{}_s g^{rs})g^{ip}g^{jq}, \tag{3.4}$$

then the spacetime metric Lagrangian

$$\mathcal{L} \equiv \sqrt{-{}^4g}\,{}^4R = -g_{ij}\pi^{ij}{}_{,0} - NR^0 - N_iR^i - 2\left(\pi^{ij}N_j - \frac{1}{2}\pi N^i + N^{|j}\sqrt{{}^3g}\right)_{,i}, \tag{3.5}$$

where we denote with a semicolon (;) the usual covariant derivative in the full spacetime and with a bar (|) a spatial covariant derivative, and

$$R^0 \equiv -\sqrt{{}^3g}\left[{}^3R + \frac{1}{{}^3g}\left(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}\right)\right], \tag{3.6}$$

$$R^i \equiv -2\pi^{ij}{}_{|j}, \tag{3.7}$$

$$\pi \equiv \pi^i_i. \tag{3.8}$$

Eq. (3.6) is the Hamiltonian constraint whereas Eq. (3.7) the momentum constraint. In fact, since the last term in Eq. (3.5) only contributes a "surface" term to the metric action $S \propto \int \mathcal{L}\, d^4x$, if spacetime extends to infinity it can be taken as giving a negligible contribution.

Upon taking variations with respect to the lapse and shift provides the constraint equation $R^0 = 0$ and $R^i = 0$ and then the lapse and shift themselves can be freely specified, reflecting the fact that coordinates systems can be freely specified in both space and time.

Since g_{ij} is a strictly positive definite tensor, in our recent paper [12] we proposed to use affine variables in place of the canonical variables g_{ij} and π^{ij} in order to cure such *unholonomous* constraint. We then introduce a "dilation" conjugate variable $\pi^i_j = g_{kj}\pi^{ik}$. This classical *momentric* (a name that is the combination of momentum and metric and was invented by John Klauder) tensor and the spatial metric tensor become the new basic canonical affine variables. By doing so and recalling that $g^{ij}{}_{|k} = 0$ we reach to the following classical Lagrangian

$$\mathcal{L} = -g_{ij}\pi^{ij}{}_{,0} - NR^0 - N_iR^i, \tag{3.9}$$

$$R^i = -2g^{ik}\pi_k{}^j{}_{|j}, \tag{3.10}$$

$$R^0 = \frac{1}{\sqrt{{}^3g}}\left[\pi^i_j\pi^j_i - \frac{1}{2}\pi^2\right] - \sqrt{{}^3g}\,{}^3R. \tag{3.11}$$

where we dropped the gradient term in the Lagrangian since it gives no contribution to the classical action

$$S = \int_0\int_\Omega \{-g_{ij}\pi^{ij}{}_{,0} - NR^0 - N_iR^i\}\, d(ct)\, d^3\mathbf{x}. \tag{3.12}$$

where Ω is the region of space and time starts from the beginning at $t = 0$.

In Affine Quantization (AQ) we promote the two canonical affine variables g_{ij} and π^i_j to operators $\hat{g}_{ij}$ and $\hat{\pi}^i_j$ and write the corresponding affine semiclassical (including just the terms up to order $\hbar$ in the $\hbar \to 0$ limit) Lagrangian $\mathcal{L}'$ using the commutation relations between the spatial metric operator and the momentric operator (these are given, for example, in Ref. [12] and derived again in the Appendix A).

IV. PATH INTEGRAL FORMULATION OF STATISTICAL GRAVITY

> [...] e il suon di lei. Così tra questa
> immensità s'annega il pensier mio:
> e il naufragar m'è dolce in questo mare.
>
> *Giacomo Leopardi*
> *L' Infinito*

Then the action for Einstein's theory of general relativity is one for a particular field theory where the field is the metric tensor $g_{\mu\nu}(x)$ a symmetric tensor with 10 independent components, each of which is a smooth function of

4 variables. We will indicate all these components with the notation $\{g\}(x)$. We will also work in euclidean time $x^0 \equiv ct \to ict$ so that the metric signature becomes $(+ + ++)$.

The thermal average of an observable $\mathcal{O}[\{g\}(x)]$ will then be given by the following expression [3]

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{O}[\{g\}(x)] \exp(-\upsilon S') \, \mathcal{D}^{10}\{g\}(x)}{\int \exp(-\upsilon S) \, \mathcal{D}^{10}\{g\}(x)}, \tag{4.1}$$

so that $\langle 1 \rangle = 1$. Here S' is the affine action

$$S' = \int_0^\beta \int_\Omega \left\{ \frac{1}{2\kappa} \mathcal{L}' + \mathcal{L}_F N \sqrt{{}^3 g} \right\} d(ct) \, d^3\mathbf{x}, \tag{4.2}$$

$1/\upsilon$ is a positive constant of dimension of energy times length, $ct \in [0, \beta[$ where $\beta = 1/\tilde{k}_B \tilde{T}$, $\tilde{k}_B$ is a Boltzmann constant of dimensions of one divided by length and by degree Kelvin, and $\tilde{T}$ an *effective* temperature in degree Kelvin (which can be made a field [3], $\tilde{T}(\mathbf{x})$). Since the thermal average involves taking a trace we must have $g_{\mu\nu}(ct + \beta, \mathbf{x}) = g_{\mu\nu}(ct, \mathbf{x})$. We will also require periodic spatial boundary conditions on the finite volume $\Omega \subset \mathrm{I\!R}^3$ which is the closest thing to a formal thermodynamic limit. As usual we will use $\mathcal{D}^{10}\{g\}(x) \equiv \prod_x d^{10}\{g\}(x)$ and the functional integrals will be calculated on a lattice using the path integral Monte Carlo (PIMC) method [20]. Moreover we will choose $d^{10}\{g\}(x) \equiv \prod_{\mu \le \nu} dg^{\mu\nu}(x)$ where the 10-dimensional space of the 10 independent components of the symmetric metric tensor is assumed to be flat.

The determination of $\mathcal{L}'$ looks like a formidable task that needs to take care of the commutation relations among the spatial metric and the momentric operators but it seems to be necessary to overcome the numerical singularities that may arise from the *geometrical unholonomous constraint* of having a strictly positive definite spatial metric. Here we are thinking of the possible loss of ergodicity in the PIMC as its paths wander through and explore the accessible region delimited by the sharp constraints which can be variously intricate. We see AQ as a way to smooth out the geometrical constraints so to recover ergodicity and be able to sample the whole relevant region efficiently.

V. CONCLUSIONS

In this short paper we present a plausible representation (realization) of the FEBB defined in Ref. [3]. This requires the use of the ADM 3+1 splitting and the AQ procedure. We just lay down the representation but without finding its explicit form which would require a rather formidable calculus where one needs to deal with commutation relations among tensorial objects. We believe that a Monte Carlo algorithm may lose ergodicity in the presence of sharp constraints which AQ can otherwise smooth out.

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Appendix A: Commutators between the spatial metric and the momentric

We start from the Poisson brackets (at fixed time) between the two canonical variables g_{ij} and π^{ij}:

$$\{g_{ij}(x), g_{kl}(x')\} = 0, \tag{A1}$$

$$\begin{aligned}\{g_{ij}(x), \pi^{kl}(x')\} &= \frac{\delta g_{ij}(x)}{\delta g_{mn}(x'')} \frac{\delta \pi^{kl}(x')}{\delta \pi^{mn}(x'')} \\ &= \frac{1}{2} \delta^3(x - x'') \delta^3(x' - x'') \delta^k_m \delta^l_n [\delta^m_i \delta^n_j + \delta^m_j \delta^n_i] \\ &= \frac{1}{2} \delta^3(\mathbf{x} - \mathbf{x}') [\delta^k_i \delta^l_j + \delta^l_i \delta^k_j],\end{aligned} \tag{A2}$$

$$\{\pi^{ij}(x), \pi^{kl}(x')\} = 0, \tag{A3}$$

where in the second equation we used the symmetry of the metric tensor to write $g_{ij} = [g_{ij} + g_{ji}]/2$ and δ^3 is a three dimensional Dirac delta function.

We then find the Poisson brackets between the two canonical affine variables g_{ij} and $\pi_i^j = g_{ik}\pi^{kj}$:

$$\begin{aligned}
\{g_{ij}(x), \pi_k^l(x')\} &= \{g_{ij}(x), g_{kn}(x')\pi^{nl}(x')\} \\
&= g_{kn}(x')\{g_{ij}(x), \pi^{nl}(x')\} \\
&= \frac{1}{2}\delta^3(\mathbf{x}-\mathbf{x}')[\delta_j^l g_{ki}(x) + \delta_i^l g_{kj}(x)], \qquad \text{(A4)}\\
\{\pi_i^j(x), \pi_k^l(x')\} &= \{g_{in}(x)\pi^{nj}(x), g_{km}(x')\pi^{ml}(x')\} \\
&= g_{km}\pi^{nj}\{g_{in}(x), \pi^{ml}(x')\} - g_{in}\pi^{ml}\{g_{km}(x'), \pi^{nj}(x)\} \\
&= \frac{1}{2}\delta^3(\mathbf{x}-\mathbf{x}')[\delta_i^l \pi_k^j(x) - \delta_k^j \pi_i^l(x)]. \qquad \text{(A5)}
\end{aligned}$$

And in the end we pass to operator commutators, promoted from the Poisson brackets $\{\ldots,\ldots\} \to [\ldots,\ldots]/(i\hbar)$. After being smeared with suitable test functions, the result is that both the metric and the momentric tensors can be made self-adjoint operators (for example choosing for the momentric $(\hat{g}_{ik}\hat{\pi}^{jk} + \hat{\pi}^{jk}\hat{g}_{ik})/2$), and the metric operators will satisfy the required positivity requirements.

AUTHOR DECLARATIONS

Conflict of interest

The author has no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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