Formalism for the Matter Density Matrix in an ADM $3+1$ foliation of spacetime

Riccardo Fantoni*

Università di Trieste, Dipartimento di Fisica, strada Costiera 11, 34151 Grignano (Trieste), Italy

(Dated: August 12, 2026)

We propose to solve the problem of the statistical physics of a quantum many body fluid within the framework of general relativity working on the phase space of the particles instead of the metric. We give a formal expression for the Matter Density Matrix (MaDM) that is amenable to be evaluated through the path integral Monte Carlo method. We propose this as the simplest route to an estimation of the necessary general relativity corrections to the equilibrium and stability of a white dwarf.

Keywords: White dwarf;Jellium;Path Integral;Monte Carlo;General Relativity;Thermal Equivalence Principle

CONTENTS

I. Introduction 1

II. Matter density matrix 2

III. Conclusions 3

Author declarations 4
- Conflicts of interest 4
- Data availability 4
- Funding 4

References 4

I. INTRODUCTION

White dwarfs [1] are the dense remnants of stars, composed mainly of carbon and oxygen nuclei, which contain electrons, protons and neutrons and are supported against collapse induced by gravity by the electron degeneracy pressure due to Pauli exclusion principle. They remain white dwarfs till gravity overcomes electron degeneracy pressure, crushing electrons into protons to form neutrons through β^+ decay.

Therefore the simplest model for the interior of a white dwarf is that of *Jellium*. This is a system of pointwise electrons immersed in a uniform neutralizing background. The density of the electron gas is so high that the Fermi temperature is much higher than the temperature of the star and the electron gas can be considered degenerate. Chandrasekar [2–5] model of a white dwarf equilibrium structure didn't take into account the effects of general relativity. First Tolman [6], and later Oppenheimer and Volkoff [7] proposed a way to treat the hydrodynamic equations for the equilibrium and stability of a star within the framework of general relativity.

In order to make some progress towards a more realistic equation of state it is necessary to drop the assumptions of an ideal and perfect electron gas. This has been accomplished with various mean field theories. Since for a white dwarf the Wigner-Seitz radius is small, $0.0003 < r_s < 0.01$ the corrections due to the Coulomb interaction will be small. The main effect of the electrostatic corrections is to give smaller radii and larger central densities compared with Chandrasekhar models of the same mass. As described in §2.4 of the book [1] a first electrostatic correction arises because the positive charges are not uniformly distributed in the gas but are concentrated in individual nuclei. A more accurate equation of state is needed for the strongly correlated plasma at high temperature and high density in the classical regime or at low temperature and low density in the quantum regime. Taking linearly into account charge screening in the electron-ion plasma a better approximation can be obtained from the Debye-Hückel theory in the classical regime or the Thomas-Fermi (and Thomas-Fermi-Dirac or Hartree-Fock) theory in the quantum regime. An

* riccardo.fantoni@scuola.istruzione.it

even more accurate but non linear theory of screening is offered by Vlasov theory in the classical regime or Lindhard (also known as random phase approximation) theory in the quantum regime [8]. Apart from these mean field theories, perturbative [9] or integral equation [10] approaches are also available.

More progress towards an exact solution of the model of the plasma is otherwise possible employing numerical computer experiments like path integral Monte Carlo [11] at finite non zero temperature or diffusion Monte Carlo [12, 13] at zero temperature. In order to take into account the effects of general relativity in the equation of state to be used within the Tolman-Oppenheimer-Volkoff equilibrium it is then necessary to solve a path integral Monte Carlo experiment for an electron or an electron-ion plasma within general relativity.

We can then proceed following two distinct routes, either we treat the quantum particles as living in a classical (non-quantum) spacetime or classical (non-quantum) particles living in a quantum spacetime [14]. In this letter we propose to use the former route.

This welding of *causality*, inherent in general relativity, and *casuality*, inherent in the Monte Carlo method, is expected to offer an operative approach to the problem of (quantum) statistical gravity of many body liquids [14–18].

II. MATTER DENSITY MATRIX

In Ref. [14] we discussed a thermal equivalence principle that should hold when dealing with a statistical physics theory of a many body fluid within the framework of general relativity: Einstein field equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ suggest that the statistical properties *of spacetime*, determined by the Einstein-Hilbert action [15–17], and the statistical properties *of matter* moving in it, determined solely by the right hand side of Einstein equations, the stress energy tensor, should be equivalent, one should be the shadow of the other. In particular when looking at the right hand side one reaches for the density matrix of the many (scalar) particle system

$$\hat{\rho} = \exp\left(-\int_{\partial\Omega^L} \hat{T}^{\alpha\beta}\,\beta_\alpha\,dS_\beta\right), \tag{2.1}$$

that is Eq. (5.1) in Ref. [14]. In this equation we use the hat to denote the corresponding operator, $\partial\Omega^L$ is an arbitrary spacelike hypersurface bounding the $4L$–volume Ω^L occupied by the L particles, and β_α is a 4–vector such that $\sqrt{\beta_\alpha\beta^\alpha} = 1/k_BT$ with k_B Boltzmann constant and T the absolute temperature.

Within Arnowitt, Deser, and Misner (ADM) $3+1$ foliation of spacetime [17] we can evaluate the integral at the exponent of Eq. (2.1) choosing a hypersurface orthogonal to the time component so that $dS_0 = \prod_{\mu=1}^{L}\sqrt{{}^3g(\boldsymbol{r}_\mu)}\,d\boldsymbol{r}_\mu$ where ${}^3g = \det\{g_{ij}\} > 0$ with g_{ij} the metric on the spatial components and $\boldsymbol{r}_\mu$ the *shifted* spatial positions of the μ particle, with $d\boldsymbol{r}_\mu = d\boldsymbol{x}_\mu + \boldsymbol{N}dt_\mu$. Here $\vec{\boldsymbol{x}}_\mu = (\boldsymbol{x}_\mu, t_\mu)$ is the 4–vector event for particle μ and $\boldsymbol{N}$ is the *shift* in the ADM foliation

$$||g_{\alpha\beta}|| = \begin{pmatrix} -(N^2 - N^iN_i) & N_i \\ N_i & g_{ij} \end{pmatrix}, \tag{2.2}$$

where N is the *lapse* and N^i the ith spatial component of the shift (see Eq. (4) in [17]). [1] We will generally use a Greek index to denote one of the 4 spacetime components and a Roman index to denote one of the 3 spatial components. A bold face symbol denotes its status as a three dimensional vector.

By the definition of the stress-energy tensor

$$\int \hat{T}^{00}\,dS_0 = \hat{\mathscr{H}}_L, \tag{2.3}$$

where $\hat{\mathscr{H}}_L$ is the Hamiltonian operator of the L particles. For a fluid invariant under translations and rotations [2] this is the only stress-energy tensor component entering the density matrix in Eq. (2.1) [19].

Assuming $\beta_0 = 1/k_B\tilde{T}$ constant in the chosen constant time section of spacetime, we reach the following expression

$$\hat{\rho} = \exp\left(-\beta_0\hat{\mathscr{H}}_L\right). \tag{2.4}$$

The coordinates, $R = \{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_L\}$, representation of the density matrix (2.4) can then be expanded into a path integral [11]

$$\rho(R, R'; \beta_0) = \int \rho(R, R_1; \tau)\cdots\rho(R_{M-1}, R'; \tau)\prod_{k=1}^{M-1} d^{3N}R_k, \tag{2.5}$$

[1] For example for Schwarzschild metric outside a spherical shell the shifts are zero $N^2 = (1 - R_s/r)$ and $g_{rr} = (1 - R_s/r)^{-1}, g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2\sin^2\theta$ with $R_s = 2\mathscr{M}$ the Schwarzschild radius with $\mathscr{M}$ the mass of the star inside the spherical shell, by Birkhoff theorem.

[2] Here we mean invariant under the relative killing vector or isometry, i.e. a transformation that leaves the metric tensor unchanged.

where $\tau = \beta_0/M$ is a small imaginary *timestep*, and the *primitive approximation* [11] to the high temperature, $M \gg 1$, density matrix

$$\rho(R, R'; \tau) = \frac{\prod_{\mu=1}^N \sqrt{^3g(\boldsymbol{r}_\mu)}}{(4\pi\lambda\tau)^{3N/2}} \exp\left[-\sum_{\mu=1}^N g_{ij}(\boldsymbol{r}_\mu)(r^i_\mu - {r'}^i_\mu)(r^j_\mu - {r'}^j_\mu)/4\lambda\tau - \tau\mathscr{U}_N(R)\right], \tag{2.6}$$

where $\mathscr{H}_N = \mathscr{K}_N + \mathscr{U}_N$ with $\mathscr{U}_N$ the potential energy of the N particles and we assumed *non relativistic* [14] particles with a kinetic energy $\mathscr{K}_N = -\lambda \sum_{\mu=1}^N \nabla^2_\mu$ with $\lambda = \hbar^2/2m$. Fully relativistic particles with a dispersion relation $\epsilon(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$ with $\boldsymbol{p} = -i\hbar\nabla$ are *non-local* since the Taylor expansion of ϵ in $\boldsymbol{p}$ has ever increasing powers of $\boldsymbol{p}$. In order to restore locality is therefore necessary to imagine for each particle the presence of its *antiparticle* [20] with negative energy $\bar{\epsilon}(\boldsymbol{p}) = -\sqrt{\boldsymbol{p}^2 + m^2}$. In Eq. (2.6) we use Einstein convention to tacitly assume a sum over repeated indexes. That equation means that at each timeslice $t_k = k\tau$, for $k = 1, 2, \ldots, M-1$, we consider the section of spacetime at constant imaginary time and construct the representation of the high temperature density matrix on that spatial foil. Eq. (2.6) reduces to the usual expression in Euclidean space [11]. It has been found that one may need corrections due to the curvature of space and to the van Vleck determinant [21, 22]. In the most general case one would like to use self consistently the metric g_{ij} obtained solving Einstein field equations for the particular snapshot of the particles configuration at the given timeslice.

In the high temperature regime one recovers classical statistical mechanics on a curved space. For $M = 1$ and $R \approx R'$

$$\rho(R, R'; \tau) \propto \prod_{\mu=1}^N \sqrt{^3g(\boldsymbol{r}_\mu)} \exp\left[-\tau\mathscr{U}_N(R)\right], \tag{2.7}$$

where $\tau = \beta_0$ is assumed to be small.

We have previously proposed [23] to consider the $\sqrt{^3g}$ factors as an external potential

$$U_N(R) = -\frac{1}{\tau} \ln\left(\prod_{\mu=1}^N \sqrt{^3g(\boldsymbol{r}_\mu)}\right) + \ldots, \tag{2.8}$$

where the dots stand for additional corrections due to the space curvature and the van Vleck determinant [21, 22].

The classical limit (2.7) has been used in various spatial dimensions to describe the properties of Coulomb systems. These offer some exact analytic solutions in one dimension [24, 25], in two dimensions [26], and at a special value of the coupling constant in two dimensions on various surfaces: The plane [27], the cylinder [28], the sphere [29], the pseudosphere [30], the Flamm paraboloid [31, 32]. [3]

On the other hand, the path integral (2.5) with the primitive approximation (2.6) has been treated for example in Refs. [21, 22] and applied in various geometries to various physical systems. For the example of the sphere treated with the Monte Carlo method [33] see Refs. [34–36].

III. CONCLUSIONS

In order to extract the statistical mechanics properties of a many (scalar) particle fluid within general relativity, we chose to work on the right hand side of Einstein field equations, building a density matrix from the stress-energy tensor through a path integral where on each imaginary timeslice one selects a constant time foil of the $3+1$ ADM spacetime foliation.

This Matter Density Matrix (MaDM) should follow as a shadow the equivalent Metric Density Matrix (MeDM) constructed from the Einstein-Hilbert action, which gives a statistical mechanics of spacetime [17]. It is our impression that among the two equivalent routes to statistical physics this last one offers a mathematically much more cumbersome approach [37]. Nonetheless the use of the Markov stochastic processes, inherent to the Metropolis algorithm [38], to measure a structural or thermodynamic observable of the many body fluid should make no distinction among the two routes. In fact the Metropolis algorithm is very simple and powerful; it can be used to sample essentially any probability distribution function regardless of analytic complexity in any number of dimensions.

---

[3] In $d+1$ spacetime with $d \leq 2$ the Weyl conformal tensor vanishes and the Riemann tensor is completely determined by its trace, the Ricci tensor. Contracting Einstein field equations in vacuum imply a vanishing scalar curvature, the trace of Ricci tensor. Then Einstein field equations reduce to the vanishing of the Ricci tensor and consequently to the vanishing of the whole Riemann tensor. It is only in $d > 2$ that general relativity looks like a theory of the gravitational interaction. Nonetheless Flamm paraboloid is an interesting surface of non constant curvature where the Coulomb plasma still admits an exact analytic solution.

AUTHOR DECLARATIONS

Conflicts of interest

None declared.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Funding

None declared.

[1] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars. The Physics of Compact Objects* (John Wiley & Sons Inc, New York, 1983).

[2] S. Chandrasekhar and E. A. Milne, The Highly Collapsed Configurations of a Stellar Mass, Monthly Notices of the Royal Astronomical Society **91**, 456 (1931).

[3] S. Chandrasekhar, The Density of White Dwarf Stars, Phil. Mag. **11**, 592 (1931).

[4] S. Chandrasekhar, The Maximum Mass of Ideal White Dwarfs, Astrophys. J. **74**, 81 (1931).

[5] R. Fantoni, White-dwarf equation of state and structure: the effect of temperature, J. Stat. Mech. , 113101 (2017).

[6] R. C. Tolman, Static Solutions of Einstein's Field Equations for Spheres of Fluid, Phys. Rev. **55**, 364 (1939).

[7] J. R. Oppenheimer and G. M. Volkoff, On Massive Neutron Cores, Phys. Rev. **55**, 374 (1939).

[8] N. H. March and M. P. Tosi, *Coulomb Liquids* (Academic Press Inc Ltd, London, 1984).

[9] M. Gell-Mann and K. A. Bruckner, Correlation Energy of an Electron Gas at High Density, Phys. Rev. **106**, 364 (1957).

[10] K. S. Singwi, M. P. Tosi, R. H. Land, and A. Sjölander, Electron Correlations at Metallic Densities, Phys. Rev. **176**, 589 (1968).

[11] D. M. Ceperley, Path integrals in the theory of condensed Helium, Rev. Mod. Phys. **67**, 279 (1995).

[12] D. M. Ceperley and B. J. Alder, Ground State of the Electron Gas by a Stochastic Method, Phys. Rev. Lett. **45**, 566 (1980).

[13] R. Fantoni, Radial distribution function in a diffusion Monte Carlo simulation of a Fermion fluid between the ideal gas and the Jellium model, Eur. Phys. J. B **86**, 286 (2013).

[14] R. Fantoni, Many Body in General Relativity: A thermal equivalence principle, Quantum Rep. **8**, 42 (2026).

[15] R. Fantoni, Statistical Gravity through Affine Quantization, Quantum Rep. **6**, 706 (2024).

[16] R. Fantoni, Statistical Gravity and entropy of spacetime, Stats **8**, 23 (2025).

[17] R. Fantoni, Statistical Gravity, ADM splitting, and Affine Quantization, Gravitation and Cosmology **31**, 568 (2025).

[18] R. Fantoni, *Unifying Classical and Quantum Physics - How classical and quantum physics can pass smoothly back and forth* (Springer, New York, 2025).

[19] F. Becattini, Covariant Statistical Mechanics and the Stress-Energy Tensor, Phys. Rev. Lett. **108**, 244502 (2012).

[20] F. R. S. Paul Adrien Maurice Dirac, Quantised singularities in the electromagnetic field, Proc. R. Soc. A **133**, 60 (1931).

[21] B. S. DeWitt, Dynamical Theory in Curved Spaces. I. A Review of the Classical and Quantum Action Principles, Rev. Mod. Phys. **29**, 377 (1957).

[22] L. S. Schulman, *Techniques and Applications of Path Integration* (John Wiley & Sons, Technion, Haifa, Israel, 1981) chapter 24.

[23] R. Fantoni, The density of a fluid on a curved surface, J. Stat. Mech. , P10024 (2012).

[24] A. Lenard, Exact Statistical Mechanics of a One Dimensional System with Coulomb Forces, J. Math. Phys. **2**, 682 (1961).

[25] S. F. Edards and A. Lenard, Exact Statistical Mechanics of a One Dimensional System with Coulomb Forces. II. The Method of Functional Integration, J. Math. Phys. **3**, 778 (1961).

[26] A. M. Salzberg and S. Prager, Equation of State for a Two-Dimensional Electrolyte, J. Chem. Phys. **38**, 2587 (1963).

[27] B. Jancovici, Exact Results for the Two-Dimensional One-Component Plasma, Phys. Rev. Lett. **46**, 386 (1981).

[28] P. Choquard, Helv. Phys. Acta **54**, 332 (1981).

[29] J. M. Caillol, Exact results for a two-dimensional one-component plasma on a sphere, J. Phys. (France) - Lett. **42**, L (1981).

[30] R. Fantoni, B. Jancovici, and G. Téllez, Pressures for a One-Component Plasma on a Pseudosphere, J. Stat. Phys. **112**, 27 (2003).

[31] R. Fantoni and G. Téllez, Two dimensional one-component plasma on a Flamm's paraboloid, J. Stat. Phys. **133**, 449 (2008).
[32] R. Fantoni, Two component plasma in a Flamm's paraboloid, J. Stat. Mech. , P04015 (2012).
[33] M. H. Kalos and P. A. Whitlock, *Monte Carlo Methods* (John Wiley & Sons Inc., New York, 1986).
[34] R. Fantoni, One-component fermion plasma on a sphere at finite temperature, Int. J. Mod. Phys. C **29**, 1850064 (2018).
[35] R. Fantoni, One-component fermion plasma on a sphere at finite temperature. The anisotropy in the paths conformations, J. Stat. Mech. , 083103 (2023).
[36] R. Fantoni, Path Integral Monte Carlo on a Sphere, Phys. Rev. B (submitted) (2026), `http://arxiv.org/abs/2603.22338`.
[37] J. R. Klauder, Quantization of Constrained Systems, in *Methods of Quantization*, Lecture Notes in Physics, Vol. 572, edited by H. Latal and W. Schweiger (2001) p. 143.
[38] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. M. Teller, and E. Teller, Equation of state calculations by fast computing machines, J. Chem. Phys. **1087**, 21 (1953).