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Statistical Gravity and Entropy of Spacetime

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Abstract: We discuss the foundations of the statistical gravity theory we proposed in a recent publication [Riccardo Fantoni, Quantum Reports, 6, 706 (2024)].

Keywords: general relativity; Einstein–Hilbert action; statistical physics; path integral; entropy; ergodicity

1. Introduction

We propose a new horizontal theory that brings together statistical physics and general relativity.

We adopt a statistical physics [1] foundation basis in order to determine the consistency of our theory, already put forward in Ref. [2], for a statistical gravity description.

From a philosophical point of view [3], we should investigate the mathematical issues of the existence and uniqueness of the Universe as well as some anthropic questions like the fine tuning for life in our Universe or the natures of existence. We may think that before creation it was only chaos, for which one could agree that between the two signatures of the metric of spacetime (the Euclidean and the Lorentzian), the one describing statistical physics (the Euclidean) would be the most appropriate. At creation, between the before and after, it could be that one has to deal with infinite energy densities or maybe density. From a description point of view, we are already accustomed to dealing with infinities. Here, I am addressing the evolution of a Dirac delta into a Gaussian via a diffusion process. But there are many other processes.

The key logical point in the theory we are proposing to explain the origins of gravity from a statistical approach is the connection between thermodynamics and statistical physics, made possible by the statistical concept of entropy and its derivative with respect to energy. This defines the temperature. In our statistical gravity theory, the energy content is due to matter and electromagnetic fields, and the entropy is a count of the quantum states of a quasi-closed subregion of spacetime, which can be considered closed for a period of time that is long relative to its relaxation time, with energy in a certain interval. Feynman describes this in chapter 1 of his set of lectures [4], saying "If a system is very weakly coupled to a heat bath at a given 'temperature,' if the coupling is indefinite or not known precisely, if the coupling has been on for a long time, and if all the 'fast' things have happened and all the 'slow' things not, the system is said to be in *thermal equilibrium*".

Equation (2) has long been studied by John Klauder [5], and the form chosen here is just representative and in substitution of the much more rigorous one offered by that author. Other alternative points of view are also present today [6–9].

This theory based on the mathematical properties of a Wick rotation could introduce a new view of the statistical properties of spacetime as a physical entity.

Our theory can be considered a *first step* towards a more sophisticated and dignified description of spacetime.

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Academic Editor: Wei Zhu

Received: 21 February 2025
Revised: 5 March 2025
Accepted: 12 March 2025
Published: 13 March 2025

Citation: Fantoni, R. Statistical Gravity and Entropy of Spacetime. *Stats* **2025**, *8*, 23. https://doi.org/10.3390/stats8010023

2. Gentropy

Let us define a *subregion* of a macroscopic spacetime region as a part of spacetime that is very small with respect to the whole Universe and yet macroscopic.

The subregion is not closed. It interacts with the other parts of the Universe. Due to the large number of degrees of freedom of the other parts, the state of the subregion varies in a complex and intricate way.

In order to formulate a statistical theory of gravity, we need to determine the *statistical distribution* of a subregion of a macroscopic spacetime region. We know from general relativity that each spacetime subregion has a metric, so our statistical distribution will describe the statistical properties of these metric tensors $g_{\mu\nu}$.

Since different subregions "interact" weakly among themselves,

1. It is possible to consider them as *statistically independent*; i.e., the state of a subregion does not affect the probability of the states of another subregion. If $\hat{\rho}_{12}$ is the density matrix of the subregion composed by the subregion 1 and by the subregion 2, then

$$\hat{\rho}_{12} = \hat{\rho}_1 \hat{\rho}_2, \tag{1}$$

   where $\hat{\rho}_i$ is the density matrix of the subregion i. (†† General relativity is fundamentally a classical theory, while the density matrix is inherently quantum mechanical. This apparent contradiction will be solved in our discussion leading into Section 3, when we will clarify which is the main actor that is in thermal equilibrium. As it will become clear then, we think the metric tensor itself to be in thermodynamic equilibrium at a given temperature. Of course, since the metric tensor determines the distances between events of the spacetime, then this also implies that the spacetime itself is fluctuating due to thermal agitation.)
2. It is possible to consider a subregion as closed for a sufficiently small time interval. The time evolution of the density matrix of the subregion in such an interval of time is

$$\frac{\partial}{\partial t} \hat{\rho}_i = \frac{i}{\hbar} [\hat{\rho}_i, \hat{H}_i], \tag{2}$$

   where $\hat{H}_i$ is the Hamiltonian of the quasi-closed subregion i.
3. After a sufficiently long period of time, the spacetime reaches the state of statistical equilibrium in which the density matrices of the subregions must be stationary. We must then have

$$[\prod_i \hat{\rho}_i, \hat{H}] = 0, \tag{3}$$

   where $\hat{H}$ is the Hamiltonian of the closed macroscopic spacetime. This condition is certainly satisfied if

$$[\hat{\rho}_i, \hat{H}] = 0, \tag{4}$$

   for all i.

We then find that the logarithm of the density matrix of a subregion is an additive integral of motion of the spacetime.

This is certainly satisfied if

$$\ln \hat{\rho}_i = \alpha_i + \beta_i \hat{H}_i. \tag{5}$$

In the time interval in which the subregion can be considered closed, it is possible to simultaneously diagonalize $\hat{\rho}_i$ and $\hat{H}_i$. We then find

$$\ln \rho_n^{(i)} = \alpha_i + \beta_i E_n^{(i)}, \tag{6}$$

where the probabilities $\rho_n^{(i)} = w(E_n^{(i)})$ represent the distribution function in statistical gravity.

If we consider the closed spacetime as composed of many subregions and we neglect the "interactions" among them, each state of the entire spacetime can be described by specifying the state of the various subregions. Then, the number $d\Gamma$ of quantum states of the closed spacetime corresponding to an infinitesimal interval of this energy must be the product

$$d\Gamma = \prod_i d\Gamma_i, \tag{7}$$

of the numbers $d\Gamma_i$ of the quantum states of the various subregions.

In fact, we have an uncertainty principle [10,11], ruling the two conjugated variables that are the generalized 'coordinates': the metric tensor field $g_{\mu\nu}(x)$ where x is an event of spacetime and the generalized 'momentum'. The operator $\hat{\pi}^{\mu\nu} = -i\hbar\delta/\delta g_{\mu\nu}$. As usual, we have

$$\begin{aligned} \Delta g_{\mu\nu}(x)\Delta\hat{\pi}^{\alpha\beta}(x') &\geq \frac{1}{2}|\langle[g_{\mu\nu}(x), \hat{\pi}^{\alpha\beta}(x')]\rangle| \\ &= \frac{1}{2}\hbar\delta^{(4)}(x-x')\delta^\alpha_\mu\delta^\beta_\nu, \end{aligned} \tag{8}$$

where $\langle\ldots\rangle$ denotes a vacuum expectation value, Δ indicates a standard deviation, $\delta^{(4)}$ is a Dirac delta function in four dimensions, and the other two δ are Kronecker symbols. (It has been pointed out by Prof. John R. Klauder that in the Arnowitt, Deser, and Misner [12] $3+1$ foliation scheme, it seems to be necessary to treat the path integral described in Section 3, where the imaginary time naturally splits from the space and the generalized 'coordinate' is played by the spatial components of the metric tensor field. But this subtensor must be positive-definite. Therefore, due to this anholonomous constraint, the corresponding generalized 'momentum' would cease to be a self-adjoint operator. The most elegant way to put things back in order is to use *affine quantization*, which amounts to defining a different generalized momentum, the so-called *dilation* operator $\hat{\pi}_a^b = g_{ac}\hat{\pi}^{cb}$, where we indicate with a Latin index a spatial component. Therefore, the dilation operator is made self-adjoint by construction.) Here, we are associating $g_{\mu\nu}(x)$ with the metric tensor from general relativity, included in Equation (27). A delicate point is that of a consistent description of the vacuum of general relativity, where both matter fields and the Ricci scalar vanish, for which our high-temperature density matrix of Equation (27) reduces to a functional Dirac delta.

We can then formulate the expression for the *microcanonical distribution function*, writing

$$dw \propto \delta(E - E_0) \prod_i d\Gamma_i \tag{9}$$

for the probability to find the closed spacetime in any of the states $d\Gamma$.

Let us consider a spacetime that is closed for a period of time and is long relative to its relaxation time. This implies that the spacetime is in complete statistical equilibrium.

Let us divide the spacetime region into a large number of macroscopic parts and consider one of these. Let $\rho_n = w(E_n)$ be the distribution function for such part. In order to obtain the probability $W(E)dE$ that the subregion has an energy between E and $E + dE$, we must multiply $w(E)$ by the number of quantum states with energies in this interval. Let us call $\Gamma(E)$ the number of quantum states with energies less than or equal to E. Then, the required number of quantum states with energy between E and $E + dE$ is

$$\frac{d\Gamma(E)}{dE} dE, \tag{10}$$

and the energy probability distribution is

$$W(E) = \frac{d\Gamma(E)}{dE} w(E), \tag{11}$$

with the normalization condition

$$\int W(E) dE = 1. \tag{12}$$

The function $W(E)$ has a well-defined maximum in $E = \bar{E}$. We can define the "width" ΔE of the curve $W = W(E)$ through the relation

$$W(\bar{E}) \Delta E = 1. \tag{13}$$

or

$$w(\bar{E}) \Delta \Gamma = 1, \tag{14}$$

where

$$\Delta \Gamma = \frac{d\Gamma(\bar{E})}{dE} \Delta E, \tag{15}$$

is the number of quantum states corresponding to the energy interval ΔE at $\bar{E}$. This is also called the *statistical weight* of the macroscopic state of the subregion, and its logarithm

$$S = \log \Delta \Gamma, \tag{16}$$

is the *entropy* of the subregion. The entropy cannot be negative.

We can also write the definition of entropy in another form, expressing it directly in terms of the distribution function. In fact, we can rewrite Equation (6) as

$$\log w(\bar{E}) = \alpha + \beta \bar{E}, \tag{17}$$

so that

$$\begin{aligned} S = \log \Delta \Gamma &= -\log w(\bar{E}) = -\langle \log w(E_n) \rangle \\ &= -\sum_n \rho_n \log \rho_n = -\mathrm{tr}(\hat{\rho} \log \hat{\rho}), \end{aligned} \tag{18}$$

where 'tr' denotes the trace.

Let us now consider the closed region again, and let us suppose that $\Delta\Gamma_1, \Delta\Gamma_2, \ldots$ are the statistical weights of the various subregions; then, the statistical weight of the entire region can be written as

$$\Delta \Gamma = \prod_i \Delta \Gamma_i, \tag{19}$$

and

$$S = \sum_i S_i, \tag{20}$$

the entropy is additive.

Let us consider the microcanonical distribution function again for a closed region,

$$\begin{aligned} dw &\propto \delta(E - E_0) \prod_i \frac{d\Gamma_i}{dE_i} dE_i \\ &\propto \delta(E - E_0) e^S \prod_i \frac{dE_i}{\Delta E_i} \\ &\propto \delta(E - E_0) e^S \prod_i dE_i, \end{aligned} \tag{21}$$

where $S = \sum_i S_i(E_i)$ and $E = \sum_i E_i$. Now we know that the most probable values of the energies E_i are the mean values $\bar{E}_i$. This means that the function $S(E_1, E_2, \ldots)$ must have its maximum when $E_i = \bar{E}_i$ for all i. However, the $\bar{E}_i$ are the values of the energies of the subregions that correspond to the complete statistical equilibrium of the region. We then reach the important conclusion that the entropy of a closed region in a state of complete statistical equilibrium has its maximum value (for a given energy of the region E_0).

Let us now again consider the problem of finding the distribution function of the subregion, i.e., of any macroscopic region being a small part of a large closed region. We then apply the microcanonical distribution function to the entire region. We will call the "medium" what remains of the spacetime region once the small macroscopic part has been removed. The microcanonical distribution can be written as

$$dw \propto \delta(E + E' - E_0) d\Gamma d\Gamma', \tag{22}$$

where $E, d\Gamma$ and $E', d\Gamma'$ refer to the subregion and to the "medium", respectively, and E_0 is the energy of the closed region that must equal the sum $E + E'$ of the energies of the subregion and of the medium.

We are looking for the probability w_n of one state of the region so that the subregion is in some well-defined quantum state (with energy E_n), i.e., a well-defined microscopic state. Let us then take $d\Gamma = 1$, set $E = E_n$, and integrate it with respect to Γ'

$$\begin{aligned} \rho_n &\propto \int \delta(E_n + E' - E_0) d\Gamma' \\ &\propto \int \frac{e^{S'}}{\Delta E'} \delta(E_n + E' - E_0) dE' \\ &\propto \left(\frac{e^{S'}}{\Delta E'} \right)_{E' = E_0 - E_n}. \end{aligned} \tag{23}$$

Now, we use the fact that, since the subregion is small, its energy E_n will be small with respect to E_0,

$$S'(E_0 - E_n) \approx S'(E_0) - E_n \frac{dS'(E_0)}{dE_0}. \tag{24}$$

However, we know that the derivative of the entropy with respect to the energy is $\beta = 1/k_B T$, where k_B is the Boltzmann constant and T is the temperature of the closed spacetime region (that coincides with that of the subregion with which it is in equilibrium). So, we finally reach the following result

$$\rho_n \propto e^{-\beta E_n}. \tag{25}$$

which is the *canonical distribution function*.

3. Metric Representation of the Density Matrix and Path Integral

We then reach the following expression for the density matrix of spacetime

$$\hat{\rho} \propto e^{-\beta \hat{H}}, \tag{26}$$

where $\hat{H}$ is the spacetime Hamiltonian. In the non-quantum high-temperature regime, we can let $\beta \to \beta/M$ with M a large integer. Then, for the high-temperature density matrix, we can use the usual classical limit [2,12–14]

$$\rho(g_{\mu\nu}, g'_{\mu\nu}; \tau) \propto \exp\left[-\tau \int_\Omega \left(\frac{1}{2\kappa}R + \mathcal{L}_F\right) \sqrt{{}^3g}\, d^3\mathbf{x}\right] \delta[g_{\mu\nu}(x) - g'_{\mu\nu}(x)], \tag{27}$$

where $g_{\mu\nu}(x)$ is the spacetime metric tensor, $x \equiv (ct, \mathbf{x}) = (x^0, x^1, x^2, x^3)$ is an event in space($\mathbf{x}$)time(t), $\tau = \beta/M$ is a small complex time step, R is the Ricci scalar of the spacetime subregion, $\kappa = 8\pi G c^{-4}$ is Einstein's gravitational constant (G is the gravitational constant and c is the speed of light in vacuum), Ω is the volume of space of the subregion whose spacetime is curved by the matter and electromagnetic fields due to the term $\mathcal{L}_F$, and ${}^3g$ is the determinant of the spatial block of the metric tensor. In Equation (27), the δ is a functional delta [15].

Using the Trotter formula [16], we reach the path integral expression described in Ref. [2] for the finite temperature case, where the metric tensor path wanders in the spacetime subregion made of the complex time interval $[0, \hbar\beta/c]$ with periodic boundary conditions and the spatial region Ω. The spatial region can be compact in the absence of black holes or not if any are present. In any case, it can either include its outermost frontier or not, but from a numerical point of view, it is convenient to use periodic boundary conditions there in order to simulate a thermodynamic limit so that only the frontiers around eventual black holes matter. The metric tensor 10-dimensional space is a hypertorus with $g_{\mu\nu}(ct + \beta(\mathbf{x}), \mathbf{x}) = g_{\mu\nu}(ct, \mathbf{x})$ and $g_{\mu\nu}(ct, \mathbf{x} + \boldsymbol{\xi}) = g_{\mu\nu}(ct, \mathbf{x})$. In the classical regime, when β is small, and if the periodicities along different spatial dimensions are incommensurable, i.e., ξ^i/ξ^j for $i \neq j$ cannot be written as rational numbers, then the Einstein field equations will let the metric tensor explore its phase space in a quasi-periodic fashion, so that one can use either a "molecular-" (or "hydro-") dynamics numerical simulation strategy, since the imaginary time averages equal the ensemble averages thanks to ergodicity, or a Monte Carlo numerical simulation strategy. In the quantum regime, when β is big, it is necessary to use the path integral Monte Carlo method described above.

The field theory we are approaching with our path integral method, where the main actor is the metric tensor field, may be subject to triviality problems as the ones that occur, for example when treating the φ_n^r scalar Euclidean covariant relativistic quantum field theory [5] for $r \geq 2n/(n-2)$, where n is the number of spacetime dimensions and r the integer positive power of the interaction term $g|\varphi(x)|^r$, with g being the coupling constant. It could be that just as in that scalar case, affine quantization also plays a crucial role in the tensorial case studied here. As already mentioned in parenthesis †† above, affine quantization is a necessary tool to treat a constrained theory with mathematical rigor. Moreover, it is different from the quantization proposed by Ashtekar [6], and as such, it is novel.

4. Conclusions

We provided a logical foundation for the statistical gravity horizontal theory that we recently proposed [2,12]. Our weakness in discussing Equation (2) does not reflect a weakness in the current knowledge and studies around that equation, but it is just representative of our lack of deep vertical awareness.

Our statistical theory of gravitation defines a temperature of the metric tensor that measures the distances between events in spacetime as the derivative of the energy of the metric with respect to its entropy.

Funding: This research received no external funding

Data Availability Statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest: The author has no conflicts of interest to disclose.

References

1. Landau, L.D.; Lifshitz, E.M. Statistical Physics. In *Course of Theoretical Physics*; Lifshitz, E.M., Pitaevskii, L.P., Eds.; Sykes, J.B.; Kearsley, M.J., Translators; Butterworth Heinemann, Linacre House: Oxford, UK, 1951; Volume 5.
2. Fantoni, R. Statistical Gravity through Affine Quantization. *Quantum Rep.* **2024**, *6*, 706–713. [CrossRef]
3. Ellis, G.F.R. Issues in the philosophy of cosmology. In *Philosophy of Physics*; Butterfield, J., Earman, J., Eds.; Elsevier B.V.: Amsterdam, The Netherlands, 2006; p. 1183. [CrossRef]
4. Feynman, R.P. Statistical Mechanics: A Set of Lectures. In *Frontiers in Physics*; Shaham, J., Ed.; W. A. Benjamin, Inc.: Urbana, IL, USA, 1972; Volume 36.
5. Klauder, J.R.; Fantoni, R. The Magnificent Realm of Affine Quantization: Valid results for particles, fields, and gravity. *Axioms* **2023**, *12*, 911. [CrossRef]
6. Ashtekar, A. New variables for classical and quantum gravity. *Phys. Rev. Lett.* **1986**, *57*, 2244. [CrossRef] [PubMed]
7. Rovelli, C. Statistical mechanics of gravity and the thermodynamical origin of time. *Class Quantum Grav.* **1993**, *10*, 1549. [CrossRef]
8. Banerjee, R.; Majhi, B.R. Statistical Origin of Gravity. *Phys. Rev. D* **2010**, *81*, 124006. [CrossRef]
9. Rovelli, C. General relativistic statistical mechanics. *Phys. Rev. D* **2013**, *87*, 084055. [CrossRef]
10. Károlyházy, F. Gravitation and quantum mechanics of macroscopic objects. *Nuovo Cimento* **1966**, *42*, 390. [CrossRef]
11. Maziashvili, M. Cosmological implications of Károlyházy uncertainty relation. *Phys. Lett. B* **2007**, *652*, 165. [CrossRef]
12. Fantoni, R. Statistical Gravity, ADM Splitting, and AQ 2024. Available online: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=5098498 (accessed on 10 March 2025).
13. Schulman, L.S. *Techniques and Applications of Path Integration*; John Wiley & Sons: Haifa, Israel, 1981.
14. Ceperley, D.M. Path Integrals in the Theory of Condensed Helium. *Rev. Mod. Phys.* **1995**, *67*, 279. [CrossRef]
15. Gelfand, I.M.; Fomin, S.V. *Functional Analysis*; Prentice-Hall Inc.: Englewoods Cliff, NJ, USA, 1963.
16. Trotter, H.F. On the Product of Semi-Groups of Operators. *Proc. Am. Math. Soc.* **1959**, *10*, 545. [CrossRef]