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A classical Luttinger liquid

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ABSTRACT

We propose a binary nonadditive Asakura–Oosawa-like mixture as an example for the classical limit of a Luttinger liquid. We determine the equation of state and structure of this mixture and discuss the paradoxical situation that one faces when working with a quantum liquid without a ground state. We then propose a new class of one dimensional classical fluids.

1. Introduction

In 1950 Tomonaga [1] published a theory of interacting fermions which was soluble only in one dimension with the provision that certain truncations and approximations were introduced into his operators. Nevertheless he had success in showing approximate boson-like behavior of certain collective excitations, which he identified as "phonons". Today we would denote these as "plasmons", following the work of Bohm and Pines [2]. Later, Luttinger [3] has revived interest in the subject by publishing a variant model of spinless and massless one dimensional interacting fermions, which demonstrated a singularity at the Fermi surface. Then Mattis and Lieb [4] corrected Luttinger results showing that there is a difference between very large determinants and infinitely large ones; they show that one of the important differences is the failure of certain commutators to vanish in the field-theoretic limit when common sense and experience based on finite N (here N refers to the number of particles in the field) tells us they should vanish. The idea of Luttinger to fill the "infinite sea" of negative energy levels with fictitious noninteracting fermions, in order to have a liquid which would admit a ground state, arrived 30 years after the discovery of antiparticles by Dirac [5]. A Luttinger's antiparticle is just the hole left in the infinite sea obtained by exciting one of the fictitious negative energy particles to a positive energy level. The locality requirement claimed by Dirac when he introduced the antiparticle in his single body relativistic quantum field theory becomes now a ground state existence requirement in the many body Luttinger liquid theory.

In this work we underline the Luttinger's paradox, of a many body system without a ground state, discussing the classical limit of his quantum liquid and propose a binary nonadditive Asakura–Oosawa-like mixture as a possible classical limit of a Luttinger liquid. We will then determine its equation of state and radial distribution function, exactly analytically. Of course the antiparticles are a purely quantum phenomena and there is no classical analogue for them. We therefore propose a new interesting class of one dimensional classical many body systems as those that can be obtained as the classical limit of a Luttinger liquid.

2. The model

Consider a Luttinger liquid [4]. Massless quantum indistinguishable particles of two species in one dimension. N particles of species "1" with positions at $x_n \in [0, L)$ with n = 1, ..., N and M particles of species "2" with positions at $y_m \in [0, L)$ with

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m = 1, ..., M, with periodic boundary conditions on the segment [0, L). The many particle system obeys to the following Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V} , \qquad (2.1)$$

$$\mathcal{H}_{0} = \sum_{n=1}^{N} p_{xn} - \sum_{m=1}^{N} p_{ym} , \qquad (2.2)$$

$$\mathcal{V} = 2\lambda \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V(|x_n - y_m|) .$$
(2.3)

where we adopt natural units $\hbar = c = 1$ and $p_{xn} = -i\partial/\partial x_n$ and $p_{ym} = -i\partial/\partial y_m$ are the momenta of particles of species "1" and "2" respectively. In the original Luttinger liquid model [4] the particles are fermions and obey to the Fermi–Dirac statistics. But, for the time being and for the sake of simplicity, we will assume that they are distinguishable and obey to Boltzmann statistics. Then, if we define a function W(x) as follows

$$dW(x)/dx = \operatorname{sgn}(x)V(x) , \qquad (2.4)$$

and we choose the following wave function

$$\Psi = \prod_{n=1}^{N} e^{ik_n x_n} \prod_{m=1}^{M} e^{iq_m y_m} \exp\left[i\lambda \sum_{n=1}^{N} \sum_{m=1}^{M} W(|x_n - y_m|)\right] , \qquad (2.5)$$

this is readily seen to obey the following Schrödinger's equation

$$\mathcal{H}\Psi = E\Psi, \qquad (2.6)$$

with just the unperturbed eigenvalues

$$E = \sum_{n=1}^{N} k_n - \sum_{m=1}^{M} q_m .$$
(2.7)

The wave numbers are of the form

$$k_n$$
 or $q_m = 2\pi$ integer/L, (2.8)

as required by periodic boundary conditions. So the spectrum of \mathcal{H} is the same as the one of \mathcal{H}_0 independent of the interaction $V(|x_n - y_m|)$.

In the classical limit we should choose $p_{xn} \in [0, +\infty)$ and $p_{ym} \in (-\infty, 0]$ so that the canonical partition function for the liquid at an absolute temperature $T = 1/k_B\beta$, where k_B is constant, is well defined and given by

$$\mathcal{Z}(\{N,M\},L,T) = \int_{0}^{L} dx^{N} \int_{0}^{L} dy^{M} \int_{0}^{\infty} dp_{x}^{N} \int_{-\infty}^{0} dp_{y}^{M} \frac{e^{-\beta H}}{(2\pi)^{N+M} N! M!}$$
(2.9)

$$= \int_{0}^{\infty} dx^{N} \int_{0}^{\infty} dy^{M} \frac{e^{-\rho \nu}}{(2\pi\beta)^{N+M} N! M!}$$
(2.10)

$$= \exp(-\beta A) \tag{2.11}$$

where $x^N = (x_1, \dots, x_N)$, $y^M = (y_1, \dots, y_M)$, $p_x^N = (p_{x1}, \dots, p_{xN})$, $p_y^M = (p_{y1}, \dots, p_{yM})$, and *A* is the Helmholtz free energy. Then for the ideal gas, $\mathcal{V} = 0$, we find

$$\beta A = -\ln\left[\left(\frac{L}{2\pi\beta}\right)^{N+M}\frac{1}{N!M!}\right],\qquad(2.12)$$

$$\beta P = -\frac{\partial \beta A}{\partial L} = \frac{N+M}{L} = \rho , \qquad (2.13)$$

with ρ the density and *P* the pressure of the ideal gas.

Imagine now a Hard Sphere (HS) potential with $\sigma_1 = \sigma_2 = 0, \sigma_{12} = 1$ as described in Section 4. Then, by following Eqs. (5.39)-(5.41) and (5.70) of Ref. [6], one gets for the symmetric (N = M) binary mixture, the following compressibility factor

$$Z = \frac{\beta P}{\rho} = 1 + \frac{\beta P}{1 + e^{\beta P}} .$$
(2.14)

The ideal gas equation of state (EOS) is obtained in the limits of low and high pressure. The maximum value of *Z* takes place at $\beta P \approx 1.28$ and is $Z = \beta P \approx 1.28$, i.e., at a density $\rho = 1$. As expected, the EOS is different from that of the ideal gas.

So we reached a paradoxical situation in which the classical limit does not behave as the underlying more general quantum theory.

3. Solution of the paradox

In the path integral expression for the many body density matrix (see Eq. (2.12) of Ref. [7]), Trotter's formula holds if the three operators \mathcal{H}_0 , \mathcal{V} , and \mathcal{H} are self adjoint and make sense separately, for example, if their spectrum is bounded from below. As we can immediately see from Eq. (2.7) this is not the case already for our \mathcal{H}_0 which does not admit a ground state.

First of all notice that if we restrict the wave numbers $k_n \in [0, +\infty)$ and $q_m \in (-\infty, 0]$ then the eigenvalues of Eq. (2.7) admit 0 as lower bound. But how can we achieve this restriction? Luttinger [4] proposed to assume that the particles are fermions¹ and fill the "infinite sea" of negative energy levels (i.e., all states with $k_n < 0$ and $q_m > 0$) with fictitious noninteracting particles. Only in this way one makes contact with a real massless fermion liquid with a ground state. Of course this requires working with an infinite number of particles. And the wave function $\Psi(x^N, y^M)$ becomes a functional $\Psi[\varphi_1(n), \varphi_2(m)]$ where $\varphi_\alpha(l)$ represents the position of the *l*th particle of species " α ", here $l \in \mathbb{Z}$ and $\alpha = 1, 2$. For example, we can adopt the convention of choosing for negative *l* the fictitious fermions filling the negative energy levels and for positive *l* the real fermions in the thermodynamic limit.

Of course in the classical limit the information on the indistinguishability of the particles is completely lost, the particles inevitably become labelable and distinguishable so that we have to artificially impose what may appear the additional artificial constraints $p_{xn} \in [0, +\infty)$ and $p_{ym} \in (-\infty, 0]$. And this is what we did in the previous section.

From the quantum point of view, a general and inescapable concavity theorem states that if $E_0(\lambda)$ is the ground-state energy in the presence of interactions (2.3) then

$$\partial^2 E_0(\lambda) / \partial \lambda^2 < 0 . \tag{3.1}$$

This inequality is incompatible with the previous result, viz. all E = independent of λ , which was possible only in the strange case of a system without a ground state. And this resolves the paradox.

It was first observed by Julian Schwinger [Phys. Rev. Letters **3**, 296 (1959)] that the very fact that one postulates the existence of a ground state (i.e., the filled Fermi sea) forces certain commutators to be nonvanishing even though in first quantization they automatically vanish. The "paradoxical contradictions" of which Schwinger speaks seem to anticipate the difficulties in the Luttinger model.

Luttinger mistake is well explained in Section 3 of Ref. [4] where it is clearly shown that the Fourier transform of the density operators $\rho_1(x) = \sum_n \delta(x - x_n)$ and $\rho_2(y) = \sum_m \delta(y - y_m)$, namely

$$\rho_1(p) = \sum_{n=1}^N e^{ipx_n} , \quad \rho_2(q) = \sum_{m=1}^M e^{iqy_m}$$
(3.2)

which clearly commute in the finite number of particles liquid will no longer commute in the filled Fermi sea liquid in the thermodynamic limit, since

$$[\rho_1(-p), \rho_1(p')] = [\rho_2(p), \rho_2(-p')] = \delta(p - p') \sum_{-p < k < 0} 1 = \delta(p - p') \frac{pL}{2\pi} .$$
(3.3)

where for definiteness we chose $p \ge 0$ and p' > 0 and we used Eq. (2.8) to change the sum in the above Eq. (3.3), which takes into account the presence of the fictitious particles filling the Fermi sea, into an integral in the thermodynamic limit $L \to \infty$.²

4. An Asakura-Oosawa-like model

As anticipated in Section 2 we can choose as a possible realization of a classical Luttinger liquid, a *nonadditive* two component HS mixture where $\mathcal{V} = \sum_{n=1}^{N} \sum_{m=1}^{M} \phi_{12}(|x_n - y_m|)$ with

$$\phi_{a\gamma}(r) = \begin{cases} \infty & r < \sigma_{a\gamma} \\ 0 & \text{else} \end{cases}$$
(4.1)

Here, $\sigma_{\alpha\gamma}$ is the closest possible distance between the center of a sphere of species " α " and the center of a sphere of species " γ ". If we call $\sigma_{\alpha} = \sigma_{\alpha\alpha}$ the closest distance between two spheres of the same species " α ", it is legitimate to refer to σ_{α} as the diameter of a sphere of species " α ". However, that does not necessarily mean that two spheres of different type repel each other with a distance equal to the sum of their radii. Depending on that, one can classify HS mixtures into additive or nonadditive:

- Additive HS (AHS) mixtures: $\sigma_{\alpha\gamma} = (\sigma_{\alpha} + \sigma_{\gamma})/2$ for all pairs $\alpha\gamma$.
- Nonadditive HS (NAHS) mixtures: $\sigma_{\alpha\gamma} \neq (\sigma_{\alpha} + \sigma_{\gamma})/2$ for at least one pair $\alpha\gamma$.

¹ In this case the productorials in the wave function of Eq. (2.5) must be replaced by determinants as follows det $|e^{ik_a x_l}|$ and det $|e^{iq_a y_l}|$.

² For example, for species 1, the density operator is made of the sum of two parts, one for the real particles and one for the fictitious particles filling the Fermi sea. When we apply the commutator $[\rho_1(-p), \rho_1(p')]$ to the fluid state with $p \ge 0$ and p' > 0 we find, from the cross terms, $(\sum_{k<0} - \sum_{k<-p}) \sum_{n'} \exp[i(p'-p)x_{n'}]$ where the position of the fictitious particle at x_n with momentum -p < 0 can be taken equal to the position of the real particle at $x_{n'}$ with momentum p' > 0. The Dirac delta in Eq. (3.3) can then be understood thinking at the property $\sum_{n'} \exp[i(p'-p)x_{n'}] = \delta(p-p')$ in the thermodynamic limit $N \to \infty$.



Fig. 1. Compressibility factor $Z = \beta P/\rho$ for the nonadditive nearest neighbor hard rods binary mixture described in the text. $\beta = 1/k_B T$ where k_B is Boltzmann constant, *T* is the absolute temperature, *P* is the pressure, $\rho = (N + M)/L$ is the density, $x_1 = N/(N + M)$ is the molar fraction of the particles of species "1", and $x_2 = 1 - x_1$. The continuous line shows the result for the symmetric mixture and the dashed line the one for an asymmetric one.

Furthermore, the nonadditivity is said to be negative if $\sigma_{\alpha\gamma} < (\sigma_{\alpha} + \sigma_{\gamma})/2$, while it is positive if $\sigma_{\alpha\gamma} > (\sigma_{\alpha} + \sigma_{\gamma})/2$.

As already announced in Section 2 we may realize a *classical Luttinger liquid* by choosing a binary mixture where $\sigma_1 = \sigma_2 = 0$ and $\sigma_{12} = 1 > 0$. This is a NAHS [8–13] with positive nonadditivity that describes a Widom–Rowlinson model [14,15] a particular limit of an Asakura–Oosawa (AO) like model [16,17] where not only polymers with each other but also colloids with each other are supposed not to interact. If the distance between the center of the colloids is less than σ_{12} , then the polymers are excluded from the space between the colloids. This results in the area between the colloids having a reduced concentration of polymers and therefore reduced entropy. This reduced entropy causes a force to act upon the colloids pushing them together. So that, in the notation of Section 2, we have $2\lambda V(x) = \phi_{12}(x)$ and in one dimension we should more properly talk about hard rods instead of HS.

As we can clearly see from Eqs. (2.9)–(2.11) the non ideal configuration integral is not affected by the precise form of the kinetic energy term. This will affect the free energy but not the EOS [6]. The nearest neighbor interaction condition requires that $\sigma_{\alpha\omega} \leq \sigma_{\alpha\gamma} + \sigma_{\gamma\omega} \forall \alpha, \gamma, \omega$ which in a binary mixture reduces to $2\sigma_{12} \geq \max(\sigma_1, \sigma_2)$. So we will consider this nonadditive nearest neighbor hard rods binary mixture. As shown in Ref. [6] this fluid can be solved analytically exactly, both for the thermodynamics and for the structure, in the isothermal isobaric ensemble.

Defining the molar fractions for the two species as $x_1 = N/(N + M)$ and $x_2 = M/(N + M)$ we find for the EOS

$$Z = \frac{\beta P}{\rho} = 1 + \frac{\beta P[\sqrt{1 + 4x_1 x_2 (e^{2\beta P} - 1)} - 1]}{e^{2\beta P} - 1} .$$
(4.2)

This EOS reduces to the EOS given by Eq. (2.14), in the symmetric case $x_1 = x_2 = 1/2$.

The EOS of Eq. (4.2) is shown in Fig. 1 in a (ρ , Z) plane for two choices of x_1 . The compressibility factor Z has a non monotonic dependence since it starts from Z = 1 at low density, reaches a maximum value (Z = 1.27846... in the symmetric case at $\rho = 1$) and then goes again to Z = 1 in the high density limit. This is just because the state with the highest compressibility factor is made of min(N, M) pairs of "1–2" particles (corresponding to $\rho = 1$ for the symmetric mixture) whereas in the high density limit the system segregates into N particles of species "1" and M particles of species "2", separated by only a pair of "1–2" particles.

The structure of the mixture can be extracted from Eqs. (5.73)-(5.74) of Ref. [6]. In Fig. 2 we show the partial radial distribution functions $g_{ij}(r)$ between a particle of species "*i*" and one of species "*j*", for the two mixture cases considered in Fig. 1 at a density $\rho = 0.8$, close to the maximum of *Z*. As we can see, there is a depletion of "2" particles and an abundance of "1" particles in the neighborhood of a "1" particle. We can also see a point of non derivability in the like functions at a distance $2\sigma_{12} = 2$, which is due to the fact that one has a change in behavior when a "2" particle is allowed between two "1" particles. What is interesting of this AO-like model is that the asymptotic behavior of the total partial distribution functions $h_{ij}(r) = g_{ij}(r) - 1$ is always monotonic with no oscillations!³

More generally we can say that the binary classical mixture limit of the Luttinger liquid must have non interacting like particles, i.e. $\phi_{\alpha\alpha} = 0$ for $\alpha = 1, 2$. Moreover, the classical mixture is known to admit an exact analytic solution [6] if in addition only *nearest neighbor* interactions are possible, i.e.

• a particle is *impenetrable* to a particle of different species, i.e. $\phi_{12}(r) = \infty$ for r < I;

 $^{^{3}}$ It can be shown that the dominant non zero pole of the Laplace transform of the partial radial distribution function (see Eqs. (5.37) in Ref. [6]) is the only real one. I thank Ana M. Montero for giving empirical evidence for this property.



Fig. 2. Partial radial distribution functions (like, g_{11} and g_{22} , and unlike, g_{12}) as functions of r = |x| for the nonadditive nearest neighbor hard rods binary mixture described in the text at a density $\rho = 0.8$. Of course $g_{12}(r)$ vanishes for $r < \sigma_{12} = 1$. We show the result for the two mixture already considered in Fig. 1. Clearly in the symmetric mixture $g_{11} = g_{22}$.

• the pair interactions have *finite* range, i.e. $\phi_{12}(r) = 0$ for $r \ge \mathcal{R}$;

• $\mathcal{R} \leq \mathcal{I}$.

Therefore we conclude that only if the 1–2 interaction is of hard rod type (i.e., $\mathcal{R} = \mathcal{I} = 1$) in the classical Luttinger fluid can we expect that interactions are restricted to nearest neighbor.⁴

5. Conclusions

We propose a realization of a classical Luttinger liquid and present its equation of state and structure. We discuss the Luttinger paradox and its solution which determines, through the classical limit, a new class of classical one dimensional fluids. We then discuss a case in which this class admits an exact analytic solution for the thermodynamics and the structure.

Declaration of competing interest

Our work proposes a new class of interesting classical one dimensional liquids. These are the ones that are obtained as the classical limit of the Luttinger liquid.

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Data availability

Data will be made available on request.

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⁴ In the *sticky limit*, since $\mathcal{R} \to 1$, it is not that evident that non nearest neighbor interactions are present, but intuitively one would say that this is still the case because if $\mathcal{R} - 1$ is not strictly zero (even though we take the limit $\mathcal{R} \to 1$ at the end) you already have interactions beyond nearest neighbor. One would expect the sticky limit to be very close to a situation where $\mathcal{R} - 1$ is very small but finite and the temperature is very low.

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