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Scaled affine quantization of φ_3^{12} is nontrivial

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We prove through path integral Monte Carlo that the covariant Euclidean scalar field theory, φ_n^r, where r denotes the power of the interaction term and $n = s + 1$ with s as the spatial dimension and 1 adds imaginary time, such that $r = 12$, $n = 3$ can be acceptably quantized using scaled affine quantization (AQ) and the resulting theory is nontrivial, unlike what happens using canonical quantization which finds it trivial.

Keywords: Covariant euclidean scalar field theory; affine quantization; path integral Monte Carlo; renormalizability

1. Introduction

Covariant Euclidean scalar field quantization, henceforth denoted φ_n^r, where r is the power of the interaction term and $n = s + 1$ with s the spatial dimension and 1 adds imaginary time, such that $r < 2n/(n-2)$ can be treated by canonical quantization (CQ), while models such that $r \geq 2n/(n-2)$ are trivial.[1–5] However, there exists a different approach called affine quantization (AQ)[6,7] that promotes a different set of classical variables to become the basic quantum operators and it allows to correctly quantize such models[8–15] which appear then to be nontrivial. In this work, we show with the aid of a path integral Monte Carlo (MC) analysis, that one of the special cases where $r > 2n/(n-2)$, specifically the case $r = 12$, $n = 3$, can be acceptably quantized using scaled AQ. This work complements the previous one[8] where the same analysis was carried out in the unscaled version. Such unscaled version was later found to have some shortcoming like a diverging vacuum expectation value of the field in the continuum limit.[10] In the subsequent work,[11] we discovered that a simple scaling procedure would cure such a divergence, even if this was done for a complex field. It was then not obvious whether the necessary scaling procedure would keep the affinely quantized real field theory nontrivial. In this work, we will answer affirmatively to this question. The same analysis has been carried out for the φ_4^4 theory in Ref. 16.

2. Affine Version of the Field-Theory

Our quantum covariant Euclidean relativistic field theory model has a standard Hamiltonian given by

$$H[\pi,\varphi] = \int \left\{ \frac{1}{2}\left[\pi^2(x) + \sum_{\mu=1}^{s}\left(\frac{\partial\varphi(x)}{\partial x_\mu}\right)^2 + m^2\varphi^2(x)\right] + g\varphi^r(x)\right\} d^s x, \tag{1}$$

where s denotes the number of spatial coordinates and x_0 is the time. The momentum field $\pi(x) = \partial\phi(x)/\partial x_0$ and the canonical action is $S = \int H\, dx_0$.

Next, we introduce the affine field $\kappa(x) \equiv \pi(x)\varphi(x)$, with $\varphi(x) \neq 0$ and modify the classical Hamiltonian to become[7,17,18]

$$H'[\kappa,\varphi] = \int \left\{ \frac{1}{2}\left[\kappa(x)\varphi^{-2}(x)\kappa(x) + \sum_{\mu=1}^{s}\left(\frac{\partial\varphi(x)}{\partial x_\mu}\right)^2 + m^2\varphi^2(x)\right] + g\varphi^r(x)\right\} d^s x. \tag{2}$$

In an AQ, the operator term $\widehat{\kappa}(x)\varphi^{-2}(x)\widehat{\kappa}(x) = \widehat{\pi}^2(x) + \hbar^2(3/4)\delta^{2s}(0)\varphi^{-2}(x)$ which leads to an extra "3/4" potential[19] term. So that the new affine action will formally read

$$S'[\varphi] = \int \left\{ \frac{1}{2}\left[\sum_{\mu=0}^{s}\left(\frac{\partial\varphi(x)}{\partial x_\mu}\right)^2 + m^2\varphi^2(x)\right] + g\varphi^r(x) + \frac{3}{8}\hbar^2\frac{\delta^{2s}(0)}{\varphi^2(x)+\epsilon}\right\} d^n x, \tag{3}$$

where $\epsilon > 0$ is a parameter used to regularize the "3/4" extra term. In the $g \to 0$ limit, this model remains different from a free-theory due to the new $(3/8)\hbar^2\delta^{2s}(0)/[\phi^2(x)+\epsilon]$ interaction term.

In order to explain the extra "3/4" potential term, we use the fact that the operator corresponding to the affine field κ will be the dilation operator $\widehat{\kappa} = (\widehat{\pi}\widehat{\varphi} + \widehat{\varphi}\widehat{\pi})/2$, where the regularized basic quantum Schrödinger operators are given by $\widehat{\varphi}(x) = \varphi(x)$ and $\widehat{\pi}(x) = -i\hbar\delta_{\varphi(x)} = -i\hbar\delta/\delta\varphi(x)$ so that the commutator $[\widehat{\varphi}(x), \widehat{\pi}(y)] = i\hbar\delta^s(x-y)$, where $\delta^s(x)$ is an s-dimensional Dirac delta function since $\delta_{\varphi(x)}\varphi(y) = \delta^s(x-y)$. Multiplying this by $\widehat{\varphi}$ we find $[\widehat{\varphi}, \widehat{\varphi}\widehat{\pi}] = [\widehat{\varphi}, \widehat{\pi}\widehat{\varphi}] = [\widehat{\varphi}, \widehat{\kappa}] = i\hbar\delta^s\widehat{\varphi}$ which is only valid for $\varphi \neq 0$. Then $\widehat{\kappa} = -i\hbar\{\delta_{\varphi(x)}[\varphi(x)] + \varphi(x)\delta_{\varphi(x)}\}/2 = -i\hbar\{\delta^s(0)/2 + \varphi(x)\delta_{\varphi(x)}\}$. Now, for $\varphi(x) \neq 0$, we will have that AQ sends $\widehat{\pi}^2(x)$ to

$$\begin{aligned}\widehat{\kappa}(x)\varphi^{-2}(x)\widehat{\kappa}(x) &= -\hbar^2\{\delta^s(0)/2 + \varphi(x)\delta_{\varphi(x)}\}\varphi^{-2}(x)\{\delta^s(0)/2 + \varphi(x)\delta_{\varphi(x)}\}\\ &= \hbar^2(3/4)\delta^{2s}(0)\varphi^{-2}(x) - \hbar^2\delta^2_{\varphi(x)}\\ &= \hbar^2(3/4)\delta^{2s}(0)\varphi^{-2}(x) + \widehat{\pi}^2(x).\end{aligned} \tag{4}$$

3. Lattice Formulation of the Field Theory

The theory considers a real scalar field φ taking the value $\varphi(x)$ on each site of a periodic, hypercubic, n-dimensional lattice of lattice spacing a, our ultraviolet cutoff and periodicity $L = Na$. The affine action for the field is then $S' = \int H' dx_0$, with $x_0 = ct$, where c is the speed of light constant and t is imaginary time, and H' is the Hamiltonian. The lattice formulation of the AQ field theory used in Eq. (3) is

$$S'[\varphi]/a^n \approx \frac{1}{2}\left\{\sum_{x,\mu} a^{-2}[\varphi(x) - \varphi(x+e_\mu)]^2 + m^2 \sum_x \varphi(x)^2\right\} + \sum_x g\varphi(x)^r + \frac{3}{8}\sum_x \hbar^2 \frac{a^{-2s}}{\varphi(x)^2 + \epsilon}, \tag{5}$$

where e_μ is a vector of length a in the $+\mu$ direction and the factor a^{-2s} in the effective interaction term due to the AQ procedure, namely $\frac{3}{8}\hbar^2\delta^{2s}(0)/(\varphi(x)^2+\epsilon)$, stems from the discretization of the Dirac delta function, and diverges in the continuum limit.

In order to solve such divergence,[16] we apply the scaling $\varphi \to a^{-s/2}\varphi, g \to a^{(r-2)s/2}g, \epsilon \to a^{-s}\epsilon$ to the action (5) above, in order to suppress the a^{-2s} factor, which diverges in the continuum limit, at the price of having a field of dimensions $a^{1/2}$. As a consequence of our scaling, the action becomes

$$S'[\varphi]/a^{n-s} \approx \frac{1}{2}\left\{\sum_{x,\mu} a^{-2}[\varphi(x) - \varphi(x+e_\mu)]^2 + m^2 \sum_x \varphi(x)^2\right\} + \sum_x g\varphi(x)^r + \frac{3}{8}\sum_x \hbar^2 \frac{1}{\varphi(x)^2 + \epsilon}. \tag{6}$$

In this work, we are interested in reaching the continuum limit by taking Na fixed and letting $N \to \infty$ at fixed volume L^s and absolute temperature $T = 1/k_B L$ with k_B the Boltzmann's constant.

4. MC Results

We performed the path integral MC[20–22] calculation for the AQ field theory previously done in Ref. 8 for the case $r = 12, n = 3$ using now the scaling $\varphi \to a^{-s/2}\varphi, g \to a^{5s}g, \epsilon \to a^{-s}\epsilon$, which brings to using the lattice formulation for the action of Eq. (6). In particular we calculated the renormalized coupling constant g_R and mass m_R defined in Eqs. (4.3) and (4.5) of Ref. 8, respectively.

For each N and g, we adjusted the bare mass m in such a way to maintain the renormalized mass approximately constant $m_R \approx 3$ to within a few percent (in all cases less than 5%).[1] Differently from our previous study[8] with the unscaled version of the affine field theory we did not need to choose complex m in order to fulfill this constraint. Moreover, the needed m turned out to be independent from g. Then we measured the renormalized coupling constant g_R defined in Refs. 8 and 9 for various values of the bare coupling constant g at a given small value of the lattice spacing

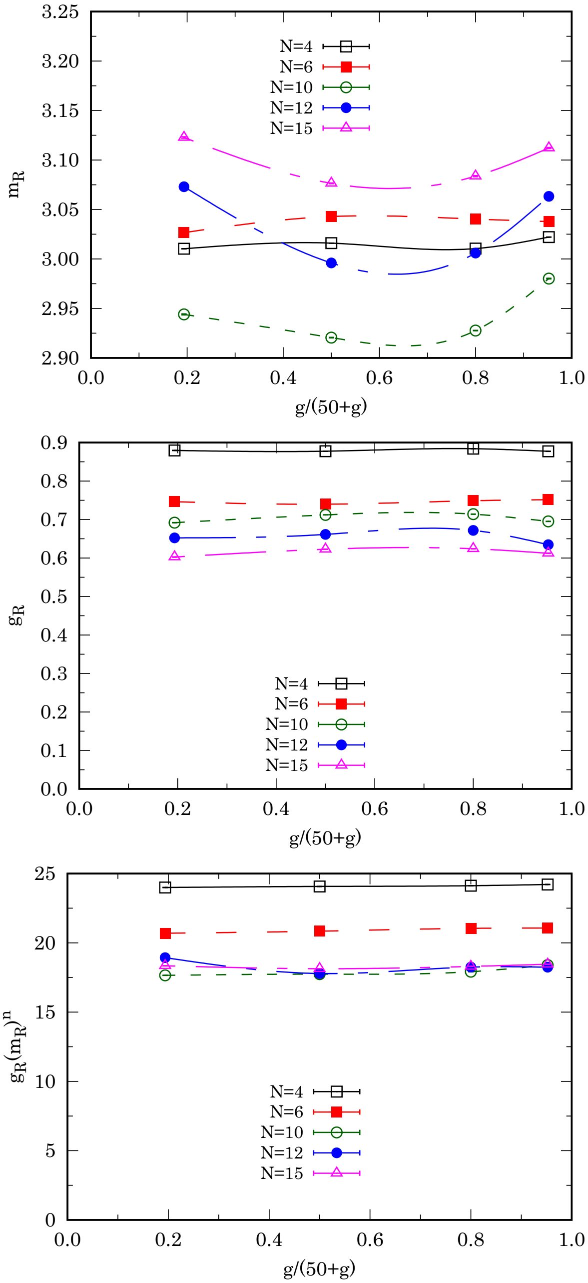

Fig. 1. (Color online) We show the renormalized mass $m_R \approx 3$ (top panel), the renormalized coupling constants g_R (central panel) and $g_R m_R^n$ (bottom panel) for various values of the bare coupling constant g at decreasing values of the lattice spacing $a = 1/N$ ($N \to \infty$ continuum limit) for the scaled affine φ_3^{12} covariant euclidean scalar field theory described by the action in Eq. (6) for $r = 12$, $n = 3$. The statistical errors were in all cases smaller than the symbols used. The lines connecting the simulation points are just a guide for the eye.

$a = 1/N$ (this corresponds to choosing an absolute temperature $k_BT = 1$ and a fixed volume $L^2 = 1$). The results are shown in Fig. 1 for the scaled affine action (6) in natural units $c = \hbar = k_B = 1$ and $\epsilon = 10^{-10}$ (the results are independent from the regularization parameter as long as this is chosen sufficiently small). The fixed renormalized mass constraint was not easy to implement since for each N and g we had to run the simulation several times with different values of the bare mass m in order to determine the value which would satisfy the constraint $m_R \approx 3$.

In our simulations, we always used 3×10^7 MC steps (which took about one week of computer time for the $N = 15$ case). We estimated that it took roughly 10–50% of each run in order to reach equilibrium from the arbitrarily chosen initial field configuration, for each set of parameters. We needed longer equilibration times for bigger N. Our MC simulations use the Metropolis algorithm[20,21] to calculate the required N^n multi-dimensional integrals. The simulation is started from the initial condition $\phi = 0$. One MC step consisted in a random displacement of each one of the N^n components of ϕ as follows: $\phi \to \phi + (\eta - 1/2)\delta$, where η is a uniform pseudo random number in $[0, 1]$ and δ is the amplitude of the displacement. Each one of these N^n moves is accepted if $\exp(-\Delta S') > \eta$ where $\Delta S'$ is the change in the action due to the move (this can be efficiently calculated considering how the kinetic part and the potential part change by the displacement of a single component of ϕ) and rejected otherwise. The amplitude δ is then chosen in such a way to have acceptance ratios as close as possible to 1/2 and is kept constant during the evolution of the simulation.

These results should be compared with the results of Fig. 1 of Ref. 8 where the same calculation was done for the canonical version of the field theory. As we can see from our present figure, contrary to Fig. 1 of Ref. 8, the renormalized coupling constant $g_R(m_R)^3$ of the scaled affine version remains far from zero in the continuum limit when the ultraviolet cutoff is removed ($\mathrm{Na} = 1$ and $N \to \infty$) for all values of the bare coupling constant g. Here, unlike in the canonical version used in Ref. 8, the diminishing space between higher N curves is a pointer toward a non-free ultimate behavior as $N \to \infty$ at fixed volume. Moreover as one can see the $N = 15$ results for the renormalized coupling fall above the ones for $N = 12$.

During our simulations, we kept under control also the vacuum expectation value of the field which in all cases was found to vanish in agreement with the fact that the symmetry $\varphi \to -\varphi$ is preserved. We can then say that the scaled system is profoundly different from the unscaled one previously treated in Ref. 8 where a diverging value of the expectation value of the field was found as a result of the broken symmetry. This is ultimately due to the fact that the scaling procedure avoids a diverging width of the infinite repulsive barrier at $\varphi = 0$ in the continuum limit and this makes possible the crossing of $\varphi = 0$ by the random walk.

5. Conclusions

In conclusion, we performed a path integral MC study of the properties (mass and coupling constant) of the renormalized covariant Euclidean scalar field theory φ_3^{12}

quantized through scaled AQ. As already pointed out in Ref. 11 where the complex field was allowed to rotate around the potential barrier at $\varphi = 0$, therefore producing a vanishing field expectation value, we here observe that due to the used scaling on the real field, its vacuum expectation value and the two-point function are well defined in the continuum limit and not diverging like what we observed in Ref. 8 without the scaling. More importantly, we are still able to show that, unlike what happens for the theory quantized through CQ, the renormalized coupling constant $g_R(m_R)^3$ does not tend to vanish in the continuum limit, when the ultraviolet cutoff is removed at fixed volume. This is a strong indication that AQ is indeed able to render renormalizable classical field theories which would be otherwise non-renormalizable when treated with CQ because of asymptotic freedom.

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