Journal of Statistical Mechanics:
Theory and Experiment

PAPER: QUANTUM STATISTICAL PHYSICS, CONDENSED MATTER, INTEGRABLE SYSTEMS

White-dwarf equation of state and structure: the effect of temperature

To cite this article: Riccardo Fantoni *J. Stat. Mech.* (2017) 113101

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

PAPER: Quantum statistical physics, condensed matter, integrable systems

White-dwarf equation of state and structure: the effect of temperature

Riccardo Fantoni

Dipartimento di Fisica, Università di Trieste, strada Costiera 11, 34151 Grignano (Trieste), Italy
E-mail: rfantoni@ts.infn.it

Received 2 July 2017
Accepted for publication 17 September 2017
Published 6 November 2017

Online at stacks.iop.org/JSTAT/2017/113101
https://doi.org/10.1088/1742-5468/aa9339

Abstract. We study the effect of having a finite temperature on the equation of state and structure of a white dwarf. In order to keep the treatment as general as possible we carry out our discussion for ideal quantum gases obeying both the Fermi–Dirac and the Bose–Einstein statistics even though we only use the results for the free electron gas inside a white dwarf. We discuss the effect of temperature on the stability of the star and on the Fermi hole.

Keywords: quantum gases

1742-5468/17/113101+17$33.00

Contents

1. Introduction

A *white dwarf* below the regime of neutron drip, at mass densities less than 4×10^{11} g cm^{-3}, are stars that emit light of a white color due to their relatively high surface temperature of about 10^4 K. Because of their small radii R, luminous white dwarfs, radiating away their residual thermal energy, are characterized by much higher effective temperatures, T, than normal stars even though they have lower luminosities (which vary as R^2T^4). In other words, white dwarfs are much 'whiter' than normal stars, hence their name [1–3].

The life of white dwarfs begins when a star dies, they are, therefore, *compact objects* [4]. Star death begins when most of the nuclear fuel has been consumed. A white dwarfs has about one solar mass $M_\odot$ with characteristic radii of about 5000 km and mean densities of around 10^6 g cm^{-3}. They are no longer burning nuclear fuel and are slowly cooling down as they radiate away their residual thermal energy.

They support themselves against gravity by the pressure of cold electrons, near their degenerate, zero temperature state. In 1932 Landau [5] presented an elementary explanation of the equilibrium of a white dwarf that had been previously discovered by Chandrasekhar in 1931 [6–8], building, on the formulation of the Fermi–Dirac statistics in August 1926 [9] and the work of Fowler in December 1926 [10], on the role of the *electron degeneracy pressure* to keep the white dwarf from gravitational collapse. Landau's explanation can be found in section 3.4 of the book of Shapiro and Teukolsky [4], and fixes the equilibrium maximum mass of the white dwarf at $M_{\rm max}\sim 1.5M_\odot$, whereas Chandrasekhar's result was $M_{\rm Ch}=1.456M_\odot$ for completely ionized matter made of elements with a ratio between mass number and atomic number equal to 2. Strictly speaking, one would have a matter made of a fluid of electrons and a fluids of nuclei. In the work of Chandrasekhar the fluid of electrons is treated as an ideal gas where the electrons are not interacting among themselves and the nuclei, thousands times heavier, are neglected.

https://doi.org/10.1088/1742-5468/aa9339

Despite their high surface temperature, these stars are still considered cold, however, because on a first approximation temperature does not affect the equation of state of its matter. White dwarfs are described as faint stars below the main sequence in the Hertzsprung–Russell diagram. In other words, white dwarfs are less luminous than main-sequence stars of corresponding colors. While slowly cooling, the white dwarfs change in color from white to red and finally to black. White dwarfs can be considered as one possibility for the final stage of stellar evolution since they are considered static over the lifetime of the Universe.

White dwarfs were established in the early 20th century and have been studied and observed ever since. They comprise an estimated 3% of all the stars of our galaxy. Because of their low luminosity, white dwarfs (except the very nearest ones) have been very difficult to detect at any reasonable distance and that is why there was very little observational data supporting the theory at the time of them being discovered. The companion of Sirius, discovered in 1915 by Adams [11, 12], was among the earliest to become known. The cooling of white dwarfs is not only a fascinating phenomenon but in addition offers information of many body physics in a new setting since the circumstances of an original star cannot be built up in a laboratory. Moreover, the evolution and the equation of state for white dwarfs can be useful on Earth, providing us with more understanding of matter and physics describing the Universe.

In this work, we discuss how the Chandrasekhar analysis at zero temperature should be changed in order to take into account the effect of having a quantum ideal gas at finite (non-zero) temperature. For the sake of generality we will treat in parallel the case of the Fermi and the Bose ideal gases. Only the Fermi case is appropriate for the description of the white dwarf interior made of ionized matter characterized by a sea of free cold electrons (as Chandrasekhar did, we will neglect the Coulomb interaction between the electrons and disregard the nuclei in order to keep the treatment analytically solvable. We will also use Newtonian gravity to study the star stability disregarding general relativistic effects). At the typical surface temperature and density of a white dwarf the momentum thermal average fraction of particles having momentum $\hbar\mathbf{k}$ and a full relativistic dispersion relation ($\mathcal{C}_k/\mathcal{C}_0$ where $\mathcal{C}_k$ is given by equation (2.25) below) varies appreciably over a k range that is a fraction of 0.933[1] of the k range where it is different from zero. So we generally expect the effect of temperature to play a role in the behavior of the ideal quantum gas. We will pursue our analysis for both the thermodynamic properties: as the validity of the various polytropic adiabatic equation of state as a function of density, and for the structural properties, such as the Fermi hole.

The paper is organized as follows: in section 2 we review the thermodynamic properties of the ideal quantum gases at finite temperatures. This section contains three subsections, in the first one, section 2.1, we discuss the importance of a full relativistic treatment at high densities, in the second one, section 2.2, we discuss the onset of quantum statistics as the star collapses, and in the third one, section 2.3, we present the revised Chandrasekhar analysis. In section 3 we present our study of the structure of the ideal quantum gases at finite temperature and in the full relativistic regime.

[1] This value will get smaller as the star cools down in view of equation (2.20) and will eventually become close to zero as the momentum thermal average fraction approaches a step function.

J. Stat. Mech. (2017) 113101

2. The thermodynamics of the ideal quantum gas

We want to find the thermodynamic grand potential of a system of many free fermions or bosons with a rest mass m in thermodynamic equilibrium at an inverse temperature $\beta = 1/k_{\mathrm{B}}T$.

The Hamiltonian of the system is

$$\mathcal{H} = \sum_i (-\hbar^2 c^2 \Delta_i + m^2 c^4)^{1/2}, \tag{2.1}$$

with Δ the Laplacian and c the speed of light.

Assuming the many particles are distinguishable (Boltzmannons) the density matrix operator, $\hat{\rho}_{\mathrm{D}}$, satisfies the Bloch equation

$$\frac{\partial \hat{\rho}_{\mathrm{D}}(\beta)}{\partial \beta} = -\mathcal{H}\hat{\rho}_{\mathrm{D}}(\beta), \tag{2.2}$$

$$\hat{\rho}_{\mathrm{D}}(0) = \mathcal{I}, \tag{2.3}$$

where $\mathcal{I}$ is the identity operator. The solution of equation (2.2) in coordinate representation $\mathbf{R} = (\mathbf{r}_1, \ldots, \mathbf{r}_N)$, where $\mathbf{r}_i$ is the position of the ith spinless particle in three-dimensional space, has the following solution

$$\rho_{\mathrm{D}}(R_0, R_1; \beta) = \langle R_0|\mathrm{e}^{-\beta\mathcal{H}}|R_1\rangle = \int \frac{\mathrm{d}\mathbf{K}}{(2\pi)^{3N}} \mathrm{e}^{-\mathrm{i}\mathbf{K}\cdot(\mathbf{R}_0-\mathbf{R}_1)} \mathrm{e}^{-\beta\sum_i(\hbar^2c^2\mathbf{k}_i^2+m^2c^4)^{1/2}}, \tag{2.4}$$

where $\mathbf{K} = (\mathbf{k}_1, \ldots, \mathbf{k}_N)$ and $\mathbf{R}_n = (\mathbf{r}_1^n, \ldots, \mathbf{r}_N^n)$. A very simple calculation yields the propagator ρ_{D} in closed form. The result can be cast in the following form

$$\rho_{\mathrm{D}} = \prod_i \mathcal{R}(\mathbf{r}_i{}^1, \mathbf{r}_i{}^0), \tag{2.5}$$

where $\mathcal{R}$ in one dimension is

$$\mathcal{R}_{1\mathrm{d}}(\mathbf{r}^1, \mathbf{r}^0) = \frac{mc^2\beta}{\pi\Psi^{1/2}} K_1\left(\frac{mc}{\hbar}\Psi^{1/2}\right), \tag{2.6}$$

where $\Psi = (\mathbf{r}^1 - \mathbf{r}^0)^2 + (\hbar c\beta)^2$ and K_ν is the familiar modified Bessel functions of order ν. In three dimensions we thus find

$$\begin{aligned}\mathcal{R}(\mathbf{r}^1, \mathbf{r}^0) &= -\frac{1}{2\pi|\mathbf{r}^1 - \mathbf{r}^0|}\frac{\mathrm{d}\mathcal{R}_{1\mathrm{d}}(\mathbf{r}^1, \mathbf{r}^0)}{\mathrm{d}|\mathbf{r}^1 - \mathbf{r}^0|} \\ &= \frac{mc^2\beta}{4\pi^2\Psi^{3/2}}\left[\frac{mc}{\hbar}\Psi^{1/2}K_0\left(\frac{mc}{\hbar}\Psi^{1/2}\right) + 2K_1\left(\frac{mc}{\hbar}\Psi^{1/2}\right) + \frac{mc}{\hbar}\Psi^{1/2}K_2\left(\frac{mc}{\hbar}\Psi^{1/2}\right)\right].\end{aligned} \tag{2.7}$$

Note that for the non-relativistic gas, when $\mathcal{H} = -\lambda\sum_i \Delta_i$, ρ_{D} would have been the usual Gaussian $\Lambda^{-3N}\mathrm{e}^{-(\mathbf{R}_1-\mathbf{R}_0)^2/4\lambda\beta}$, with $\lambda = \hbar^2/2m$ and $\Lambda = \sqrt{4\pi\beta\lambda}$, the de Broglie thermal wavelength.

Taking care of the indistinguishability of the particles we can describe a system of bosons and fermions with spin $s = (g-1)/2$ through density matrices, $\hat{\rho}_{\mathrm{B,F}}$, which are obtained from the distinguishable one opportunely symmetrized or antisymmetrized, respectively. The corresponding grand canonical partition functions can then be found through a standard procedure [13] from $\Theta_{\mathrm{B,F}} = \mathrm{e}^{-\beta\Omega_{\mathrm{B,F}}} = \sum_{N=0}^{\infty} Z_{\mathrm{B,F}}^{N} \mathrm{e}^{N\mu\beta}$ where $Z_{\mathrm{B,F}}^{N} = \mathrm{e}^{-\beta F_{\mathrm{B,F}}^{N}}$ is the trace of $\hat{\rho}_{B,F}$. Here $\mu = (\ln z)/\beta$ is the chemical potential, F is the Helmholtz free energy, and Ω is the grand thermodynamic potential.

If V is the volume occupied by the system of particles, the pressure is given by $P = -\Omega/V$, and the average number of particles, $N = nV = -z\partial\beta\Omega/\partial z$, where n is the number density. We find for bosons

$$\beta P = \frac{gm^2c}{2\pi^2\beta\hbar^3}\sum_{\nu=1}^{\infty}\frac{z^\nu}{\nu^2}K_2(\beta mc^2\nu), \tag{2.8}$$

$$n = \frac{gm^2c}{2\pi^2\beta\hbar^3}\sum_{\nu=1}^{\infty}\frac{z^\nu}{\nu}K_2(\beta mc^2\nu), \tag{2.9}$$

and for fermions

$$\beta P = \frac{gm^2c}{2\pi^2\beta\hbar^3}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}z^\nu}{\nu^2}K_2(\beta mc^2\nu), \tag{2.10}$$

$$n = \frac{gm^2c}{2\pi^2\beta\hbar^3}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}z^\nu}{\nu}K_2(\beta mc^2\nu). \tag{2.11}$$

Clearly in the zero temperature limit ($\beta \to \infty$) these reduce to (see section 2.3 of [4] and our appendix)

$$P = \frac{g}{2}\frac{mc^2}{\lambda\!\!\!^{-3}}\phi(x), \tag{2.12}$$

$$n = \frac{g}{2}\frac{x^3}{3\pi^2\lambda\!\!\!^{-3}}, \tag{2.13}$$

$$\phi(x) = \frac{1}{8\pi^2}\left[x\sqrt{1+x^2}\left(\frac{2}{3}x^2-1\right)+\ln\left(x+\sqrt{1+x^2}\right)\right], \tag{2.14}$$

where $\lambda\!\!\!^{-} = \hbar/mc$, with m the electron mass, is the electron Compton wavelength.

We can then introduce the polylogarithm, b_μ, of order μ and the companion f_μ function,

$$b_\mu(z) = \sum_{\nu=1}^{\infty}\frac{z^\nu}{\nu^\mu}, \tag{2.15}$$

$$f_\mu(z) = \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}z^\nu}{\nu^\mu} = -b_\mu(-z) = \left(1-2^{1-x}\right)b_\mu(z). \tag{2.16}$$

J. Stat. Mech. (2017) 113101

At finite temperatures, in the extreme relativistic case, we find for bosons

$$\beta P = \frac{g}{\pi^2(\beta\hbar c)^3} b_4(z), \tag{2.17}$$

$$n = \frac{g}{\pi^2(\beta\hbar c)^3} b_3(z), \tag{2.18}$$

where we used the property $z\mathrm{d}b_\mu(z)/\mathrm{d}z = b_{\mu-1}(z)$, and for fermions

$$\beta P = \frac{g}{\pi^2(\beta\hbar c)^3} f_4(z), \tag{2.19}$$

$$n = \frac{g}{\pi^2(\beta\hbar c)^3} f_3(z). \tag{2.20}$$

In agreement with section 61 of Landau [14]. And in the non-relativistic case, we find for bosons

$$\beta P = \frac{g}{\Lambda^3} b_{5/2}(z), \tag{2.21}$$

$$n = \frac{g}{\Lambda^3} b_{3/2}(z), \tag{2.22}$$

and for fermions

$$\beta P = \frac{g}{\Lambda^3} f_{5/2}(z), \tag{2.23}$$

$$n = \frac{g}{\Lambda^3} f_{3/2}(z), \tag{2.24}$$

in agreement with section 56 of Landau [14]. Recalling that the internal energy of the system is given by $E = -\partial \ln \Theta/\partial\beta$, we find in the extreme relativistic case $E = 3PV$ and in the non-relativistic case $E = 3PV/2$. At very low density n, and high temperature T, when $n/T^{3/2}$ is very small, $b_{3/2}(z) \approx f_{3/2}(z)$ is very small and z is also very small. In this case $b_{3/2}(z) \approx b_{5/2}(z) \approx f_{3/2}(z) \approx f_{5/2}(z) \approx z$ and we find for the quantum gas $E/V \approx (3/2)K_{\mathrm{B}}Tn$. That is, the non-relativistic classical limit. For the bosons, as the temperature gets small at fixed density, $b_{3/2}(z)$ increases (see equation (2.22)) and z gets close to 1. $b_\mu(z)$ is a monotonically increasing function of z, which is only defined in $0 \leqslant z \leqslant 1$, so the boson ideal gas must have a chemical potential less than zero. $b_{3/2}(1) = \zeta(3/2) \approx 2.612$ and $b_{5/2}(1) = \zeta(5/2) \approx 1.341$, where ζ is the Riemann zeta function. The temperature $T_{\mathrm{c}} = \frac{2\pi\hbar^2}{mk_{\mathrm{B}}}\left(\frac{n/g}{\zeta(3/2)}\right)^{2/3}$ at which $z = 1$ is called the *critical temperature* for the Bose–Einstein condensation in the non-relativistic case. For $T < T_{\mathrm{c}}$ the number of bosons with energy greater than zero will then be $N_> = N(T/T_{\mathrm{c}})^{3/2}$. The rest $N_0 = N[1 - (T/T_{\mathrm{c}})^{3/2}]$ bosons are in the lowest energy state, i.e. have zero energy. For the fermions the activity is allowed to vary in $0 \leqslant z < \infty$ and the functions $f_\mu(z)$ can be extended at $z > 1$ by using the following integral representation $f_x(z) = [\int_0^\infty \mathrm{d}y\, y^{x-1}/(\mathrm{e}^y/z + 1)]/\Gamma(x)$, where Γ is the usual gamma function.

J. Stat. Mech. (2017) 113101

Given the entropy $S = -\partial\Omega/\partial T$ we immediately see that, in both the extreme relativistic and the non-relativistic cases, S/N must be a homogeneous function of order zero in z, and that along an adiabatic process (S/N constant) we must have z constant. Then, on an adiabatic, in the extreme relativistic case, $P \propto n^{1+1/3}$, a polytrope of index 3, and in the non-relativistic case, $P \propto n^{1+2/3}$, a polytrope of index 3/2. This conclusion clearly continues to hold at zero temperature when $z \to \infty$ and the entropy is zero.

2.1. Relativistic effects at high density in a gas of fermions

The thermal average fraction of particles having momentum $\mathbf{p} = \hbar\mathbf{k}$ is given by

$$\mathcal{C}_k = \frac{g}{N}\frac{1}{\mathrm{e}^{\beta[\epsilon(k)-\mu]} - \xi} = \frac{g}{N\xi}b_0\left(\xi z\mathrm{e}^{-\beta\epsilon_k}\right), \quad V\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\mathcal{C}_k = 1, \tag{2.25}$$

where $\xi = +1, \ -1$ and 0 refer to the Bose, Fermi and Boltzmann gases, respectively.

In a degenerate ($T = 0$) Fermi gas we can define the Fermi energy as $\epsilon_{\mathrm{F}} = \mu = \sqrt{p_{\mathrm{F}}^2c^2 + m^2c^4}$, in terms of the Fermi momentum p_{F}. From equation (2.25) it follows that the thermal average fraction of particles having momentum $\mathbf{p} = \hbar\mathbf{k}$ is $\mathcal{C}_k = (g/N)\Theta[\mu - \epsilon(k)]$, where Θ is the Heaviside unit step function and $\epsilon(k) = \sqrt{\hbar^2k^2c^2 + m^2c^4}$ is the full relativistic dispersion relation. We will then have for the density

$$n = \frac{g}{h^3}\int_0^{p_{\mathrm{F}}} 4\pi p^2\,\mathrm{d}p = \frac{4\pi g}{3h^3}p_{\mathrm{F}}^3. \tag{2.26}$$

We then see immediately that at high density the Fermi momentum is also large, and as a consequence the Fermi gas becomes relativistic. By contrast, the degenerate Bose gas will undergo Bose–Einstein condensation and have all the particles in the zero energy state.

At finite temperature, from the results of the previous section, we find that since $f_\mu(z)$ is a monotonously increasing function of z then at large density n, z is also large and at fixed temperature this implies that the chemical potential μ is also large. In view of equation (2.25) this means that in the gas there are fermions of ever increasing momentum so that a relativistic treatment becomes necessary.

From equations (2.10) and (2.11) it is possible (see appendix) to extract the full relativistic adiabatic equation of state as a function of temperature and observe the transition from the low density regime to the high density extreme relativistic one. In figure 1 we show the exponent $\Gamma = \mathrm{d}\ln P/\mathrm{d}\ln n$ for the adiabatic full relativistic equation of state as a function of density. For the sake of the calculation it may be convenient to use natural units $\hbar = c = k_{\mathrm{B}} = 1$. From the figure we see how at high density (which implies high activity) $\Gamma \to 4/3$. This figure should be compared with figure 2.3 of [4] for the degenerate Fermi gas. In particular we see how at a temperature of $T = 20\,000$ K the Fermi gas can already be considered extremely relativistic at an electron number density $n \gtrsim 10^{25}$ cm^{-3}. While we know (see [4] and equations (2.12)–(2.14)) that the completely degenerate gas becomes extremely relativistic for $n \gtrsim 10^{31}$ cm^{-3}.

J. Stat. Mech. (2017) 113101

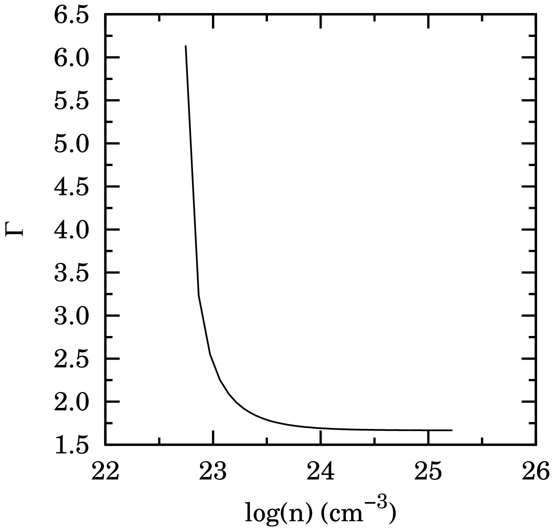

Figure 1. The exponent $\Gamma = \mathrm{d}\ln P/\mathrm{d}\ln n$ for the adiabatic full relativistic equation of state as a function of density. We chose a temperature $T = 20\,000$ K and zero entropy, $g = 2$, and m is the mass of an electron. n is in cm^{-3}.

2.2. The onset of quantum statistics

For a spherically symmetric distribution of matter, the mass interior to a radius r is given by

$$m(r) = \int_0^r \rho 4\pi r'^2\,\mathrm{d}r', \quad \text{or} \quad \frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho. \tag{2.27}$$

Here, since we are considering non-relativistic matter made of completely ionized elements of atomic number Z and mass number A, $\rho = \rho_0 = \mu_e m_u n$ is the rest mass density with $\mu_e = A/Z$ the mean molecular weight per electron and $m_u = 1.66 \times 10^{-24}$ g the atomic mass unit. If the star is in a steady state, the gravitational force balances the pressure force at every point. To derive the *hydrostatic equilibrium* equation, consider an infinitesimal fluid element lying between r and $r + \mathrm{d}r$ and having an area $\mathrm{d}A$ perpendicular to the radial direction. The gravitational attraction between $m(r)$ and the mass $\mathrm{d}m = \rho\mathrm{d}A\mathrm{d}r$ is the same as if $m(r)$ were concentrated at a point at the center, while the mass outside exerts no force on $\mathrm{d}m$. The net outward pressure force on $\mathrm{d}m$ is $-[P(r + \mathrm{d}r) - P(r)]\mathrm{d}A$, where P is the pressure. So, in equilibrium,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho}{r^2}, \tag{2.28}$$

where G is the universal gravitational constant[2].

A consequence of the hydrostatic equilibrium is the *virial theorem*. The gravitational potential energy of the star of radius R is

[2] Here we are assuming Newtonian theory of gravity. For the general relativistic stability analysis see for example section 6.9 of [4].

$$W = -\int_0^R \frac{Gm(r)}{r}\rho 4\pi r^2\,\mathrm{d}r$$
$$= \int_0^R \frac{\mathrm{d}P}{\mathrm{d}r}4\pi r^3\,\mathrm{d}r$$
$$= -3\int_0^R P4\pi r^2\,\mathrm{d}r, \tag{2.29}$$

where we have integrated by parts.

Now we assume that the gas of fermions is characterized by an adiabatic equation of state

$$P = K\rho_0^\Gamma, \quad K, \quad \Gamma = 1 + \frac{1}{n}\ \text{constants}, \tag{2.30}$$

which is also called a *polytrope* of polytropic index n. For example, for fermions in the extreme relativistic limit we find

$$K = \frac{P}{\rho^{4/3}} = \frac{\pi^{2/3}\hbar c}{g^{1/3}(\mu_e m_u)^{4/3}}\frac{f_4(z)}{f_3^{4/3}(z)}, \tag{2.31}$$

where z depends on the temperature and density and goes to infinity in the degenerate limit ($\lim_{z\to\infty} f_4(z)/f_3^{4/3}(z) = 3^{1/3}/2^{5/3}$). At the temperature and density typical of a white dwarf z is very large so the equation of state is practically indistinguishable from the one in the degenerate limit.

Calling u' the energy density of the gas, excluding the rest mass energy, we must have from the first law of thermodynamics, assuming adiabatic changes,

$$\mathrm{d}(u/\rho_0) = -P\mathrm{d}(1/\rho_0), \tag{2.32}$$

and integration leads to

$$u = \rho_0 c^2 + \frac{P}{\Gamma - 1}, \tag{2.33}$$

which gives $u' = P/(\Gamma - 1)$. Now equation (2.29) can be rewritten as

$$W = -3(\Gamma - 1)U, \tag{2.34}$$

where $U = \int_0^R u' 4\pi r^2\,\mathrm{d}r$ is the total internal energy of the star. The total energy of the star, $E = W + U$, is then

$$E = -\frac{3\Gamma - 4}{3(\Gamma - 1)}|W|. \tag{2.35}$$

If equation (2.30) holds everywhere inside the star of total mass M and constant density, then the gravitational potential energy is given by

$$W = -3\int_0^M \frac{P}{\rho}\,\mathrm{d}m(r) = -\frac{3(\Gamma - 1)}{5\Gamma}\frac{GM^2}{R}, \tag{2.36}$$

where we used $\mathrm{d}(P/\rho) = [(\Gamma-1)/\Gamma]Gm(r)\mathrm{d}(1/r)$ and integrated by parts using $\Gamma > 1$.

Without nuclear fuel, E decreases due to radiation. According to equations (2.35) and (2.36), $\Delta E < 0$ implies $\Delta R < 0$ whenever $\Gamma > 4/3$. That is, the star contracts and the gas will soon become quantum (see [4] section 3.2). Can the star contract forever, extracting energy from the infinite supply of gravitational potential energy until R goes to zero or until the star undergoes total collapse? The answer is no for stars with $M \sim M_\odot$, as is demonstrated by Chandrasekhar [15] or in the book of Shapiro and Teukolsky [4]. We will reproduce their treatments in the next section.

2.3. The Chandrasekhar limit

The hydrostatic equilibrium equations (2.27) and (2.28) can be combined to give

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho. \tag{2.37}$$

Substituting the equation of state (2.30) and reducing the result to dimensionless form with

$$\rho = \rho_c\theta^n, \tag{2.38}$$

$$r = a\eta, \tag{2.39}$$

$$a = \sqrt{\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}}, \tag{2.40}$$

where $\rho_c = \rho(r=0)$ is the central density, we find

$$\frac{1}{\eta^2}\frac{\mathrm{d}}{\mathrm{d}\eta}\eta^2\frac{\mathrm{d}\theta}{\mathrm{d}\eta} = -\theta^n. \tag{2.41}$$

This is the *Lane–Emden equation* for the structure of a polytrope of index n. The boundary conditions at the center of a polytropic star are

$$\theta(0) = 1, \tag{2.42}$$

$$\theta'(0) = 0. \tag{2.43}$$

The condition (2.42) follows directly from equation (2.38). Equation (2.43) follows from the fact that near the center $m(r) \approx 4\pi\rho_c r^3/3$, so that, using equation (2.27), $\mathrm{d}\rho/\mathrm{d}r = 0$.

Equation (2.41) can be easily integrated numerically, starting at $\eta = 0$ with the boundary conditions (2.42) and (2.43). One finds that for $n < 5$ ($\Gamma > 6/5$), the solutions decreases monotonically and have a zero at a finite value $\eta = \eta_n$: $\theta(\eta_n) = 0$. This point corresponds to the surface of the star, where $P = \rho = 0$. Thus the radius of the star is

$$R = a\eta_n, \tag{2.44}$$

while the mass is

$$\begin{aligned} M &= \int_0^R 4\pi r^2 \rho \, \mathrm{d}r \\ &= 4\pi a^3 \rho_c \int_0^{\eta_n} \eta^2 \theta^n \, \mathrm{d}\eta \\ &= -4\pi a^3 \rho_c \int_0^{\eta_n} \frac{\mathrm{d}}{\mathrm{d}\eta} \left(\eta^2 \frac{\mathrm{d}\theta}{\mathrm{d}\eta} \right) \mathrm{d}\eta \\ &= 4\pi a^3 \rho_c \eta_n |\theta'(\eta_n)|. \end{aligned} \tag{2.45}$$

Eliminating ρ_c between equations (2.44) and (2.45) gives the mass–radius relation for polytropes

$$M = 4\pi R^{(3-n)/(1-n)} \left[\frac{(n+1)K}{4\pi G} \right]^{n/(n-1)} \eta_n^{(3-n)/(1-n)} \eta_n^2 |\theta'(\eta_n)|. \tag{2.46}$$

The solutions we are particularly interested in are

$$\Gamma = \frac{5}{3}, \quad n = \frac{3}{2}, \quad \eta_{3/2} = 3.653\,75, \quad \eta_{3/2}^2 |\theta'(\eta_{3/2})| = \omega_{3/2} = 2.714\,06, \tag{2.47}$$

$$\Gamma = \frac{4}{3}, \quad n = 3, \quad \eta_3 = 6.896\,85, \quad \eta_3^2 |\theta'(\eta_3)| = \omega_3 = 2.018\,24, \tag{2.48}$$

which, as explained in section 2.1, corresponds to the low density non-relativistic case and to the high density relativistic case, respectively. Note that for $\Gamma = 4/3$, M is independent of ρ_c and hence R. We conclude that as $\rho_c \to \infty$, the electrons become more and more relativistic throughout the star, and the mass asymptotically approaches the value

$$M_{\mathrm{Ch}} = 4\pi \omega_3 \left(\frac{K}{\pi G} \right)^{3/2}, \tag{2.49}$$

as $R \to 0$. The mass limit (2.49) is called the *Chandrasekhar limit* (see equation (36) in [6], equation (58) in [16], or equation (43) in [17]) and represents the maximum possible mass of a white dwarf.

In figure 2 we show the temperature dependence of the Chandrasekar limit at $\mu_e = 2$.

For the dependence of the star mass on the central density as it develops through the various polytropes, as shown in figure 1, see for example figure 3.2 of [4]. Clearly in the high $\rho_c \to \infty$ limit we will have in the degenerate limit $z \to \infty$, from equation (2.31),

$$M \to M_{\mathrm{Ch}} = 1.456\,39 \left(\frac{2}{\mu_e} \right)^2 M_\odot, \tag{2.50}$$

where μ_e can be taken approximately equal to 2 or to 56/26, assuming that all the elements have been subject to nuclear fusion in the stable iron $^{56}_{26}\mathrm{Fe}$.

The star will not become a black hole if $R > r_{\mathrm{s}}$ (see figure 1.1 of [4]), with $r_{\mathrm{s}} = 2GM_{\mathrm{Ch}}/c^2$ the Schwarzschild radius in the Chandrasekhar limit, i.e.

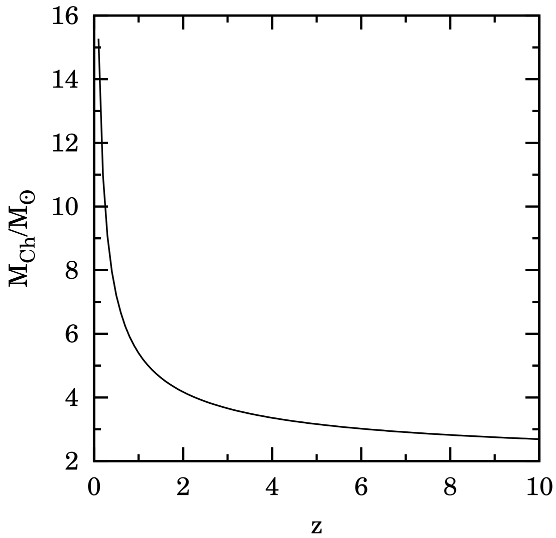

Figure 2. Temperature dependence of the Chandrasekar limit at $\mu_e = 2$. We recall that $z = e^{\beta\mu}$.

$$K < \frac{\eta_3 c^2}{2^3 \omega_3 \rho_c^{1/3}}, \tag{2.51}$$

where K is given by (2.31). This suggests that at high enough central densities the star's fate is to become a black hole. The critical central density is given in the degenerate $z \to \infty$ limit by $\bar{\rho}_c = g(\mu_e/2)^4(2.3542 \times 10^{17}$ g cm$^{-3})$ which is well above the one required for the neutron drip.

If the star has a mass lower than M_{Ch} it will not reach the Chandrasekhar limit but will remain on a polytrope with $n < 3$. If the star has a mass higher than M_{Ch} it will eventually evolve through a supernovae explosion into a more compact object as a neutron star (when electrons are captured by protons to form neutrons by β^+ decay), a quark star, or a black hole.

3. The structure of the ideal quantum gas

The radial distribution function $g(r)$ is related to the structure factor $S(k)$ by the following Fourier transform

$$n[g(r) - 1] = \frac{1}{V} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}[S(k) - 1]. \tag{3.1}$$

Taking into account that the operator of the particle number N_0 is a constant of motion, the fluctuation–dissipation theorem (see appendix of [18]) $\chi''(k,\omega) = (n\pi/\hbar)(1 - e^{-\beta\hbar\omega})S(k,\omega)$, can be solved for the van Hove function

$$S(k,\omega) = \frac{\hbar}{n\pi}[1 - \delta_{\mathbf{k}}]\frac{\chi''(k,\omega)}{1 - e^{-\beta\hbar\omega}} + \left\langle \frac{(\delta N)^2}{N} \right\rangle \delta_{\mathbf{k}}\delta(\omega), \tag{3.2}$$

J. Stat. Mech. (2017) 113101

where $\langle \ldots \rangle$ represents averaging in the grand canonical ensemble. The static structure factor $S(k) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, S(k,\omega)$ is then

$$S(k) = \frac{\hbar}{n\pi}[1 - \delta_{\mathbf{k}}] \int_0^\infty \mathrm{d}\omega\, \chi''(k,\omega) \coth\left(\frac{\beta\hbar\omega}{2}\right) + \left\langle \frac{(\delta N)^2}{N} \right\rangle \delta_{\mathbf{k}}\delta(\omega), \tag{3.3}$$

where the last term does not contribute in the thermodynamic limit [19]. We substitute (see appendix of [18])

$$\chi''(k,\omega) = N\pi \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3} \mathcal{C}_{k'} \{\delta[\hbar\omega - \Delta_{\mathbf{k}'}(\mathbf{k})] - \delta[\hbar\omega + \Delta_{\mathbf{k}'}(\mathbf{k})]\}, \tag{3.4}$$

with $\Delta_{\mathbf{k}'}(\mathbf{k}) = \epsilon(|\mathbf{k}' + \mathbf{k}|) - \epsilon(k')$, and obtain for $\mathbf{k} \neq \mathbf{0}$

$$S(k) = V \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3} \mathcal{C}_{k'} \coth\left\{\frac{1}{2}\beta[\epsilon(|\mathbf{k}' + \mathbf{k}|) - \epsilon(k')]\right\}, \quad k > 0, \tag{3.5}$$

where $\mathcal{C}_k$ denotes the thermal average fraction of particles having momentum $\hbar\mathbf{k}$ defined in equation (2.25).

For further analytical manipulation we rewrite

$$\frac{\beta}{2}[\epsilon(k) - \mu] = \ln\sqrt{\frac{g}{N\mathcal{C}_k} + \xi}. \tag{3.6}$$

One rewrites equation (3.5) changing variables first $\mathbf{k} + \mathbf{k}' \to \mathbf{k}$ and subsequently $\mathbf{k} \to -\mathbf{k}$ to find

$$S(k) = V \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3} \mathcal{C}_{|\mathbf{k}+\mathbf{k}'|} \coth\left\{\frac{1}{2}\beta[\epsilon(k) - \epsilon(|\mathbf{k} + \mathbf{k}'|)]\right\}. \tag{3.7}$$

Adding equations (3.5) and (3.7) and making use of the fact that the hyperbolic cotangent is an odd function, one finds

$$2S(k) = V \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3} (\mathcal{C}_{k'} - \mathcal{C}_{|\mathbf{k}+\mathbf{k}'|}) \coth\left\{\frac{1}{2}\beta[\epsilon(|\mathbf{k}' + \mathbf{k}|) - \epsilon(k')]\right\}. \tag{3.8}$$

Now using equation (3.6) we find

$$\begin{aligned} S(k) &= \frac{V}{2} \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3} (\mathcal{C}_{k'} - \mathcal{C}_{|\mathbf{k}+\mathbf{k}'|}) \coth\left[\ln\sqrt{\frac{g}{N\mathcal{C}_{|\mathbf{k}+\mathbf{k}'|}} + \xi} - \ln\sqrt{\frac{g}{N\mathcal{C}_{k'}} + \xi}\right] \\ &= \frac{V}{2} \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3} \left(\mathcal{C}_{k'} + \mathcal{C}_{|\mathbf{k}+\mathbf{k}'|} + \frac{2N\xi}{g}\mathcal{C}_{k'}\mathcal{C}_{|\mathbf{k}+\mathbf{k}'|}\right) \\ &= 1 + \frac{VN\xi}{g} \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3} \mathcal{C}_{k'}\mathcal{C}_{|\mathbf{k}+\mathbf{k}'|}, \quad k > 0, \end{aligned} \tag{3.9}$$

where $\coth[\ln\sqrt{x}] = (x+1)/(x-1)$ was used in the middle step. From this follows

$$\frac{1}{V}\sum_{\mathbf{k}\neq\mathbf{0}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}[S(k) - 1] = \frac{n\xi}{g}\left\{2\mathcal{C}_0 \sum_{\mathbf{k}\neq\mathbf{0}} \mathcal{C}_k \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + \left|\sum_{\mathbf{k}\neq\mathbf{0}} \mathcal{C}_k \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\right|^2\right\}, \tag{3.10}$$

J. Stat. Mech. (2017) 113101

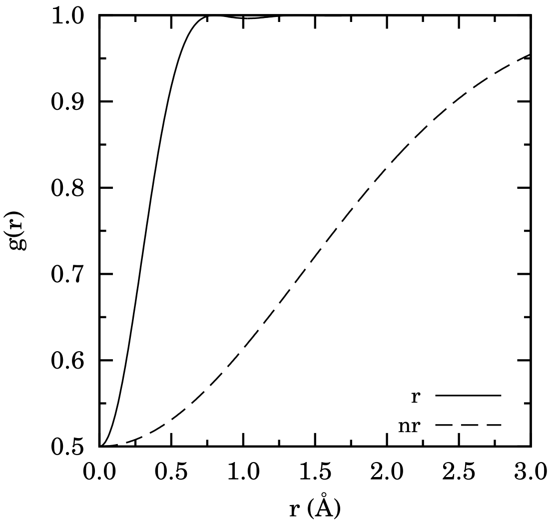

Figure 3. The radial distribution function for ideal electrons ($\xi = -1,\ g = 2$) in the relativistic and the non-relativistic cases. Here we chose $T = 20\,000$ K and $n = 1.04 \times 10^{22}$ cm^{-3} in the non-relativistic case and $n = 5.93 \times 10^{24}$ cm^{-3} in the relativistic case. r is in angstroms.

where $\mathcal{C}_0 = \delta_{\xi,1}\Theta(T_c - T)N_0/N$, with Θ the Heaviside step function, denotes the fraction of particles that occupy the zero momentum state. We then introduce the function $F(r) = \sum_{\mathbf{k}} \mathcal{C}_k e^{i\mathbf{k}\cdot\mathbf{r}}$. This assume the following forms

$$F_r(r) = \mathcal{C}_0(T) + \frac{g}{2\pi^2 n(\beta\hbar c)^2\xi}\int_0^\infty \kappa d\kappa\, b_0\left(\xi z e^{-\sqrt{\kappa^2+\beta^2 m^2 c^4}}\right)\sin\left(\frac{1}{\beta\hbar c}\kappa r\right)/r, \tag{3.11}$$

$$F_{er}(r) = \mathcal{C}_0(T) + \frac{g}{2\pi^2 n(\beta\hbar c)^2\xi}\int_0^\infty \kappa d\kappa\, b_0\left(\xi z e^{-\kappa}\right)\sin\left(\frac{1}{\beta\hbar c}\kappa r\right)/r, \tag{3.12}$$

$$F_{nr}(r) = \mathcal{C}_0(T) + \frac{2g}{\pi n\Lambda^2\xi}\int_0^\infty \kappa d\kappa\, b_0\left(\xi z e^{-\kappa^2}\right)\sin\left(\frac{2\sqrt{\pi}}{\Lambda}\kappa r\right)/r. \tag{3.13}$$

in the relativistic $\epsilon(k) = \sqrt{\hbar^2 k^2 c^2 + m^2 c^4}$, extreme relativistic $\epsilon(k) = c\hbar k$, and non-relativistic $\epsilon(k) = \lambda k^2$ cases, respectively. Inserting equations (3.9) into (3.1) we find

$$g(r) = 1 + \frac{\xi}{g}\left[F^2(r) - \mathcal{C}_0^2(T)\right]. \tag{3.14}$$

which generalizes equation (117.8) of Landau [14]. In figure 3 we show the redial distribution function for fermions in the relativistic and the non-relativistic cases. From the figure we see how the Fermi hole becomes larger in the non relativistic case at smaller number densities. Increasing the temperature by one order of magnitude (see figure 3.3 of [4]), keeping the density fixed produces a change in the radial distribution function of the order of 10^{-2}, with the Fermi hole getting smaller.

For the electron gas we should include the Coulomb interaction between the particles: the *jellium*. The radial distribution function of the jellium cannot of course be calculated exactly analytically; for a Monte Carlo simulation of the degenerate ($T = 0$)

J. Stat. Mech. (2017) 113101

jellium see, for example, [20] and for the jellium at finite temperature see, for example, [21].

Actually a more accurate result could be found by treating the white dwarf matter as a binary mixture of electrons and nuclei, which can today be done exactly with Monte Carlo simulation techniques such as the one devised in [22].

From these numerical studies one could extract a more accurate value for the constant K in the adiabatic equation of state and thus the critical central density $\bar{\rho}_{\mathrm{c}} = (\eta_3 c^2/2^3\omega_3 K)^3$.

4. Conclusions

In this work we studied the importance of temperature dependence on ideal quantum gases relevant for white dwarf interiors. Even if the temperature of the star is six orders of magnitude smaller than the Fermi energy of the electron gas inside the star, we find that the temperature effects are quite relevant at white dwarf densities and temperatures. In particular we show that the adiabatic equation of state becomes extremely relativistic, with $\Gamma = 4/3$, at densities six orders of magnitude lower than the ones required for the completely degenerate, $T = 0$, case. Even if the polytropic form of the adiabatic equation of state remains the same as that at zero temperature, the proportionality constant K changing by just a 10^{-10} relative factor between the finite temperature case and the zero temperature case, we think that an accurate analysis of the star evolution, at least at the level of the ideal electron gas approximation in the absence of nuclei, should properly take into account the temperature effects. This gives us a complete exactly solvable analytic approximation for the compact star interior at a finite temperature. We could comment that the temperature effects are smaller than the corrections necessary to take into account the Coulomb interactions between the electrons and of the presence of the nuclei, but from a calculation point of view it is still desirable to keep under control the magnitude of the temperature corrections alone. Since this can be done analytically we think that their analysis is relevant by itself.

We gave the generalization to finite temperature of all the zero temperature results used by Chandrasekhar and, in order to keep the treatment as general as possible, we studied in parallel the Fermi and the Bose gases. Clearly, only the Fermi gas results were used for the description of the ideal electron gas in the star interior.

We then studied the structure of the ideal quantum gas as a function of temperature. We found the Fermi hole for the cold electron gas in a white dwarf, which turned out to be of the order of 1 Å in the full relativistic regime at a number density of the order of $n \sim 10^{26}$ cm^{-3} and bigger in the non-relativistic regime at smaller densities and fixed temperature. The radial distribution function was also affected by the temperature and the Fermi hole gets smaller as the temperature increases at fixed density.

We also pointed out that in order to correct our result for the Coulomb interaction among the electrons and for the presence of the nuclei, it is necessary to abandon the analytic treatment in favor of a numerical simulation. We gave some relevant references for Monte Carlo methods that are important to adopt to solve this fascinating subject. These corrections to the Chandrasekhar result or to our temperature dependent treatment are important more from a philosophical point of view rather than an experimental or observational point of view. They would lead us to the exact knowledge of the properties of a mixture of electrons and nuclei at astrophysical conditions such as the ones found in white dwarfs.

Moreover, let us observe that only a general relativistic statistical physics theory would give us fully correct results for the stability of a white dwarf. But since this theory has not yet been formulated [23] we will have to wait until the theory becomes available.

Appendix. The adiabatic equation of state for a relativistic ideal electron gas at finite temperature

Using the dispersion relation $\epsilon(k)=\sqrt{\hbar^2k^2c^2+m^2c^4}$, with m the rest mass of an electron, we find the pressure and the density from,

$$\beta P = g\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\ln\left(1+z\mathrm{e}^{-\beta\epsilon(k)}\right), \tag{A.1}$$

$$n = g\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\frac{1}{\mathrm{e}^{\beta\epsilon(k)}/z+1}. \tag{A.2}$$

Integrating by parts the pressure equation and changing variable $\kappa=\beta\hbar ck$ we find

$$\beta P = \frac{g}{(\beta\hbar c)^3}\frac{1}{2\pi^2}\frac{1}{3}\int \mathrm{d}\kappa\,\frac{\kappa^3/\sqrt{\kappa^2+(\beta mc^2)^2}}{\mathrm{e}^{\sqrt{\kappa^2+(\beta mc^2)^2}}/z+1}, \tag{A.3}$$

$$n = \frac{g}{(\beta\hbar c)^3}\frac{1}{2\pi^2}\int \mathrm{d}\kappa\,\frac{\kappa^2}{\mathrm{e}^{\sqrt{\kappa^2+(\beta mc^2)^2}}/z+1}. \tag{A.4}$$

These equations are equivalent to equations (2.10) and (2.11) in the main text. Then the entropy is given by

$$S/Vk_{\mathrm{B}} = g\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\ln\left(1+z\mathrm{e}^{-\beta\epsilon(k)}\right) - g\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\frac{\ln z-\beta\epsilon(k)}{\mathrm{e}^{\beta\epsilon(k)}/z+1}. \tag{A.5}$$

On an adiabatic the entropy per particle $s=S/Nk_{\mathrm{B}}$ is constant, and from equation (A.1) it follows that

$$\beta P = g\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\frac{\ln z-\beta\epsilon(k)}{\mathrm{e}^{\beta\epsilon(k)}/z+1}+sn. \tag{A.6}$$

References

[1] Balian R and Blaizot J P 1999 *Am. J. Phys.* **67** 1189
[2] Silbar R and Reddy S 2004 *Am. J. Phys.* **72** 892
[3] Jackson C B, Taruna J, Pouliot S L, Ellison B W, Lee D D and Piekarewicz J 2005 *Eur. J. Phys.* **26** 695
[4] Shapiro S L and Teukolsky S A 1983 *Black Holes, White Dwarfs and Neutro Stars, the Physics of Compact Objects* (New York: Wiley)

J. Stat. Mech. (2017) 113101

[5] Landau L D 1932 *Phys. Z. Sowjetunion* **1** 285
[6] Chandrasekhar S 1931 *Mon. Not. R. Astron. Soc.* **91** 456
[7] Chandrasekhar S 1931 *Phil. Mag.* **11** 592
[8] Chandrasekhar S 1931 *Astrophys. J.* **74** 81
[9] Dirac P A M 1926 *Proc. R. Soc.* A **112** 661
[10] Fowler R H 1926 *Mon. Not. R. Astron. Soc.* **87** 114
[11] Adams W S 1915 *Publ. Astron. Soc. Pac.* **27** 236
[12] Adams W S 1925 *Proc. Natl Acad. Sci. USA* **11** 382
Adams W S 1925 *Observatory* **49** 88 (erratum)
[13] Feynman R P 1972 *Statistical Mechanics: a Set of Lectures* (*Frontiers in Physics* vol 36) (New York: Benjamin) (notes taken by R Kikuchi and H A Feiveson, edited by Jacob Shaham)
[14] Landau L D and Lifshitz E M 1980 *Statistical Physics, Part* I (*Course of Theoretical Physics* vol 5) 3rd edn (Portsmouth, NH: Butterworth Heinemann) (translated from the Russian by J B Sykes and M J Kearsley, edited by E M Lifshitz and L P Pitaevskii)
[15] Chandrasekhar S 1938 *An Introduction to the Study of Stellar Structure* (Chicago, IL: University of Chicago Press)
[16] Chandrasekhar S 1935 *Mon. Not. R. Astron. Soc.* **95** 207
[17] Chandrasekhar S 1983 Nobel prize lecture
[18] March N H and Tosi M P 1984 *Coulomb Liquids* (New York: Academic)
[19] Bosse J, Pathak K N and Singh G S 2011 *Phys. Rev.* E **84** 042101
[20] Fantoni R 2013 *Eur. Phys. J.* B **86** 286
[21] Militzer B, Pollock E L and Ceperley D M 2003 arXiv:0310401 [cond-mat]
[22] Dewing M and Ceperley D M 2002 *Recent Advances in Quantum Monte Carlo Methods,* II ed W A Lester *et al* (Singapore: World Scientific)
[23] Rovelli C 2013 *Phys. Rev.* D **87** 084055