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The density of a fluid on a curved surface

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Abstract. We discuss the property of the number density of a fluid of particles living on a curved surface without boundaries to be constant in the thermodynamic limit. In particular we find a sufficient condition for the density to be constant along the Killing vector field generating a given isometry of the surface, and the relevant necessary condition. We reinterpret the effect of a curvature on the fluid in a physical way as responsible for an external 'force' acting on the particles.

Keywords: rigorous results in statistical mechanics

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1. Introduction

The physics of fluids of particles living on surfaces is a well known subject in surface physics. A special role is played by low-dimensional, exactly analytically solvable fluids, as they give approximate solutions in higher dimensions and general sum rules. In the statistical mechanics of continuous fluids, those where the particles are allowed to move in a continuous space, one finds exact solutions for various Coulomb fluids [1]. For example the one-component Coulomb plasma (OCP) is exactly solvable in one dimension [2]. In two dimensions Jancovici and Alastuey [3, 4] proved that the OCP is exactly solvable analytically at a special value of the coupling constant. Since then, a growing interest in two-dimensional plasmas has led to this system being studied on various flat geometries [5]–[7] and two-dimensional curved surfaces, such as cylinders [8, 9], spheres [10]–[14], pseudospheres [15]–[17], and Flamm's paraboloids [18, 19]. Among these surfaces only the last one is of non-constant curvature. The statistical mechanics of liquids and fluids in curved spaces is a field of growing interest [20].

Here we do not restrict ourselves to those exactly solvable cases but wish to find a general property of any given fluid living on a curved surface without boundaries. A *homogeneous* fluid living on a plane (or in general a Euclidean space) is known [21] to have a *constant density*. This same conclusion holds for a (non-ideal) fluid living on a surface of constant curvature in its thermodynamic limit[1]. In this paper we will state what can be said about the constancy of the density for a fluid living on a Riemannian surface without boundaries and embeddable in the three-dimensional Euclidean space, in its thermodynamic limit. It is obvious that an ideal fluid (a gas) has a constant density on any surface, whether or not we are in the thermodynamic limit. But what can be said about a non-ideal fluid?

[1] The notion of thermodynamic limit will become clear further on in the paper.

The study of [18] showed that the OCP on a Flamm's paraboloid is indeed homogeneous. We expect this occurrence to be due to the long-range nature of the Coulomb potential and argue that it cannot hold in general for other choices of the pair potential or of the surface.

In this work we will give a physical interpretation of the curvature of the surface as an external 'force' guiding the particles of the corresponding 'flat' fluid. We will show that the Coulomb potential has to be a function of the geodesic distance between the charges and we will restrict ourselves to a definition of a fluid as one made of particles with a pair interaction potential which is a function of the geodesic distance between the two particles. We will then find a necessary and sufficient condition for the density multiplied by the square root of the determinant of the metric tensor to be constant along a certain direction. We will show how this condition holds true both for non-quantum and quantum fluids.

The paper is organized as follows: in section 2 we state the problem we want to solve at the level of non-quantum fluids; in section 3 we reformulate the problem in such way as to make explicit the physical interpretation of the curvature of the surface; section 4 is devoted to the quantum fluid formulation of the problem; section 5 is for final remarks.

2. Statement of the problem

Given a non-quantum fluid of point-like particles *living on* a surface $\mathcal{S}$ embeddable in a three-dimensional Euclidean space (note that we will not take into consideration those surfaces deriving from a Riemannian metric but not embeddable or those not deriving from a metric) and without boundaries one can define the canonical ensemble particle *number density* as [21]

$$\rho(\mathbf{q}_1) = \frac{N}{Z}\int_\Omega \mathrm{e}^{-\beta V(\mathbf{q}_1,\ldots,\mathbf{q}_N)} \prod_{i=2}^{N} \sqrt{g(\mathbf{q}_i)} \wedge_{\alpha_i=1}^{2} \mathbf{d}q^{\alpha_i}, \tag{1}$$

$$Z = \int_\Omega \mathrm{e}^{-\beta V(\mathbf{q}_1,\ldots,\mathbf{q}_N)} \prod_{i=1}^{N} \sqrt{g(\mathbf{q}_i)} \wedge_{\alpha_i=1}^{2} \mathbf{d}q^{\alpha_i}, \tag{2}$$

where N is the number of particles confined in the region Ω, $\beta = 1/k_\mathrm{B}T$ with k_B Boltzmann's constant and T the absolute temperature. The potential energy of the fluid is $V = \sum_{1\le i<j\le N} v(d(\mathbf{q}_i, \mathbf{q}_j))$ where v is the pair potential and $d(\mathbf{q}, \mathbf{q}')$ is the geodesic distance between the two points $\mathbf{q}$ and $\mathbf{q}'$. The surface is defined by a metric tensor $g_{\alpha\beta}$ so that the square of the proper length of the infinitesimal line element is given, using the usual Einstein summation convention, by $\mathbf{ds}^2 = g_{\alpha\beta}(\mathbf{q})\mathbf{d}q^\alpha \otimes \mathbf{d}q^\beta$, where $\otimes$ is the usual tensor product. We denote with $g(\mathbf{q}) = \det \|g_{\alpha\beta}(\mathbf{q})\|$ the Jacobian of the transformation from a locally flat reference frame to the local coordinates system on the surface. Here we use a coordinate basis $\{\mathbf{e}_\alpha = \partial_{q^\alpha}\}$ so that $\mathbf{q} = q^\alpha \mathbf{e}_\alpha$ and the symbol $\mathbf{d}$ stands for the exterior derivative. As usual we use superscript Greek indices for contravariant components and subscript Greek indices for covariant components, and we use a subscript Roman index to denote the (distinguishable) particle number. The symbol $\wedge$ indicates the usual wedge product. In the following we will call $\mathrm{vol}(\Omega) = \int_\Omega \sqrt{g(\mathbf{q})} \wedge_{\alpha=1}^{2} \mathbf{d}q^\alpha$ the *volume* of the region Ω.

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The problem we want to discuss is that of finding continuous transformations that leave unchanged the density $\rho(\mathbf{q})$ in the *thermodynamic limit*. Here we think of the surface $\mathcal{S}$ as an embeddable one without boundaries. And by thermodynamic limit we mean that, if $\mathcal{S}$ extends to infinity, $\mathrm{vol}(\Omega) \to \infty$ with $\overline{\rho} = N/\mathrm{vol}(\Omega)$ kept constant, or if $\mathcal{S}$ is closed, $\Omega \to \mathcal{S}$ with $\overline{\rho} = N/\mathrm{vol}(\mathcal{S})$. We want to answer the question: 'when is $\rho(\mathbf{q})$ constant on $\mathcal{S}$ in the thermodynamic limit?'.

The number density satisfies the following normalization condition

$$\int_\Omega \rho(\mathbf{q})\sqrt{g(\mathbf{q})}\wedge_{\alpha=1}^{2}\mathbf{d}q^\alpha = N = \mathrm{vol}(\Omega)\overline{\rho}. \tag{3}$$

So when the density is constant on the surface we must have $\rho = \bar{\rho}$.

3. Reinterpretation of the curvature

Choosing the coordinate basis so that $\boldsymbol{\xi} = \partial_{q^\alpha}$ is a Killing vector field [22] generating an *isometry*, then $g_{,\alpha} = 0$, where we use the usual comma convention to indicate a partial directional derivative. We know that if $\mathbf{p}$ is the momentum of a free particle on $\mathcal{S}$ then $\mathbf{p}\cdot\boldsymbol{\xi}$ is a constant of motion $p_\alpha(\mathbf{p}\cdot\boldsymbol{\xi})^{;\alpha} = 0$, where we use the usual semicolon convention to indicate a covariant derivative. An ideal gas has constant density on every surface, regardless of the curvature and of the thermodynamic limit. We thus have to worry about the term $\exp(-\beta V)$. Now, if one moves the N particles at $\mathbf{q}_1, \ldots, \mathbf{q}_N$ along the vector field $\boldsymbol{\xi}$, the geodesic distances among the system of particles will stay constant as well as the potential energy V. We then have proved that, given a Killing vector field ∂_{q^α}, then $\rho_{,\alpha} = 0$. Strictly speaking, before taking the thermodynamic limit, the domain has boundaries, and close to these one might not be able to move the particles along the Killing vector field, invalidating the conclusion near the boundary. When taking the thermodynamic limit, one needs to be able to quantify if these boundary effects will be negligible or not, and how deep they can affect the bulk of the system. This depends on the pair potential v and on the surface. In a flat space it is well known that the boundary effects are negligible (for suitable short-range potentials and for the Coulomb potential for globally neutral systems to have screening). But for a general curved surface, a proper study of what happens in the thermodynamic limit with this boundary effect is needed and it will certainly impose additional conditions on the pair potential v, and probably also on the surface, to keep valid the conclusion that $\rho_{,\alpha} = 0$. The conditions on the surface might appear, for example, in cases similar to the pseudosphere, where it has been shown that boundary effects can be of the same order of magnitude as the bulk properties (see Refs. [15]–[17]). So, additional work in this direction is needed.

This is clearly only a *sufficient* condition, but it is enough to say that on the sphere (or the plane), a surface of constant curvature [23], where $\boldsymbol{\xi} = \partial_\varphi$, with φ the azimuthal angle, the density will be constant in the thermodynamic limit. One, in fact, has that the density is constant along parallels. And this, given the symmetries of the sphere, means that the density is indeed everywhere constant over the whole sphere, with $\rho = \overline{\rho}$.

On the other hand a *necessary* condition can be expressed as follows: Say that we find a coordinate system such that, for all v, $(\sqrt{g}\rho)_{,\alpha} = 0$, then in particular for $v = 0$ we have ρ =constant and $g_{,\alpha} = 0$. For a Flamm's paraboloid [18] we can say that there certainly

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exists a fluid (at least one v) such that $(\sqrt{g}\rho)_{,r} \neq 0$, since ∂_r is not a Killing vector of the surface and $g_{,r} \neq 0$. And we know [18] that the OCP is an example.

The problem then reduces to understanding what can be said about surfaces of non-constant curvature. Note that we can as well rewrite equation (1) as follows

$$\sqrt{g(\mathbf{q}_1)}\rho(\mathbf{q}_1) = N\frac{\int_\Omega \mathrm{e}^{-\beta[V(\mathbf{q}_1,\ldots,\mathbf{q}_N)+\sum_{i=1}^N \phi(\mathbf{q}_i;\beta)]}\prod_{i=2}^N \wedge_{\alpha_i=1}^2 \mathbf{d}q^{\alpha_i}}{\int_\Omega \mathrm{e}^{-\beta[V(\mathbf{q}_1,\ldots,\mathbf{q}_N)+\sum_{i=1}^N \phi(\mathbf{q}_i;\beta)]}\prod_{i=1}^N \wedge_{\alpha_i=1}^2 \mathbf{d}q^{\alpha_i}}\ , \tag{4}$$

where $\phi(\mathbf{q};\beta) = -[\ln g(\mathbf{q})]/2\beta$ is an 'external potential'. A form which suggests, on physical grounds, a local dependence of the density on the curvature. The fluid is seen in this formulation as living on a 'flat space', the two-dimensional space determined by the local coordinates chart (q^1, q^2) used in the surface, subject to an external potential induced by the metric. This suggestive reinterpretation of the problem can sometimes lead to a wrong intuition. For example, we know that the OCP on a Flamm's paraboloid (see section 4.2.4 of [18]) has a density that is everywhere constant, even if this surface is only asymptotically flat but curved near the 'horizon', the scalar curvature being proportional to the Euclidean distance r from the origin to the power of minus three. Whereas the constancy of the density along the azimuthal direction φ has to be expected from the sufficient condition stated above, the constancy of the density along the radial r direction is not at all intuitive, even more so in the light of the discussion which follows.

For a surface with a conformal metric $g_{\alpha\beta} = \sqrt{g(\mathbf{q})}\delta_{\alpha\beta}$,[2] the scalar curvature R can be written as

$$R(\mathbf{q}) = \mathrm{e}^{\beta\phi(\mathbf{q})}\beta\Delta_{\mathrm{flat}}\phi(\mathbf{q}), \tag{5}$$

where $\Delta_{\mathrm{flat}} = \partial^2_{q^1} + \partial^2_{q^2}$ is the flat Laplace operator. The external 'force' acting on the particles due to the curvature is then $-R\exp(-\beta\phi)/\beta$. For a Flamm's paraboloid [18] the force acting on the charges turns out to be $4/[\beta s(1+s)^2]$, where $s = \sqrt{(q^1)^2 + (q^2)^2}$. As we have already mentioned above, in this case, the OCP shows a constant density on the surface. In section 3.2 we show that in general one would expect a non-constant density.

On the other hand the formulation of equation (4) suggests that $\sqrt{g}\rho$ should certainly be regarded as a more fundamental quantity than ρ.

3.1. The Coulomb pair potential

Here we want to show that the Coulomb potential between two charged particles living on a given surface $\mathcal{S}$ has to be a function of the geodesic distance between the charges [3, 8, 10, 15, 16, 18].

The Coulomb potential is defined by the Poisson equation,

$$\Delta_{\mathbf{q}} v_{\mathrm{Coul}}(\mathbf{q},\mathbf{q}') = -2\pi\delta^{(2)}(\mathbf{q},\mathbf{q}'), \tag{6}$$

where $\Delta_{\mathbf{q}}$ is the Laplace–Beltrami operator and $\delta^{(2)}(\mathbf{q},\mathbf{q}') = \delta^{(2)}(d(\mathbf{q},\mathbf{q}'))$ the Dirac delta function, in the surface $\mathcal{S}$. The Laplace-Beltrami operator is invariant to isometries. This means that if the charge at $\mathbf{q}$ and the one at $\mathbf{q}'$ are moved along the vector field of

[2] Note that the following are all surfaces of this kind: the sphere embedded in three-dimensional Euclidean space $\sqrt{g} = 4/(1+s^2)^2$, the pseudosphere embedded in three-dimensional Minkowski space $\sqrt{g} = 4/(1-s^2)^2$, the cylinder embedded in three-dimensional Euclidean space $\sqrt{g} = 1$, and a Flamm's paraboloid embedded in three-dimensional Euclidean space $\sqrt{g} = (1 + 1/s)^4$. Here $s = \sqrt{(q^1)^2 + (q^2)^2}$.

an isometry the Laplace-Beltrami operator will not change. Neglecting eventual additive functions which have a null Laplacian we must have

$$v_{\text{Coul}} = f(d(\mathbf{q},\mathbf{q}')). \tag{7}$$

For example on the sphere [10] of radius R one finds $f(x) = -\ln(2R\sin(x/2R)/L)$, with L a length scale. The conclusion of equation (7) is in agreement with Fermat's principle for light propagation [24].

3.2. The Coulomb fluid

For an open surface with a conformal metric $g_{\alpha\beta} = (\sqrt{g(s)}/s)\delta_{\alpha\beta}$, $s \in [0, +\infty[$ the Laplace-Beltrami operator can be rewritten as

$$\Delta f = \frac{s}{\sqrt{g}}\Delta_{\text{flat}} f, \tag{8}$$

where Δ_{flat} is the usual Laplace operator in flat space $(x = s\cos\varphi, y = s\sin\varphi)$. We can then introduce a complex coordinate $z = s\mathrm{e}^{\mathrm{i}\varphi}$ and the Laplacian Green's function (6)

$$\Delta_{\text{flat}} v_{\text{Coul}}((s,\varphi),(s_0\varphi_0)) = -2\pi\frac{1}{s}\delta(s-s_0)\delta(\varphi-\varphi_0) \tag{9}$$

can be solved as usual, by using the decomposition as a Fourier series. Since (6) reduces to the flat Laplacian Green's function, the solution is the standard one

$$v_{\text{Coul}}((s,\varphi),(s_0\varphi_0)) = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{s_<}{s_>}\right)^n \cos[n(\varphi-\varphi_0)] + v_0(s,s_0), \tag{10}$$

where $s_> = \max(s, s_0)$ and $s_< = \min(s, s_0)$. The Fourier coefficient for $n = 0$ has the form

$$v_0(s,s_0) = \begin{cases} a_0^+ \ln s + b_0^+ & s > s_0 \\ a_0^- \ln s + b_0^- & s < s_0 \end{cases}, \tag{11}$$

and it has to satisfy the boundary conditions that v_0 should be continuous at $s = s_0$, $a_0^+ \ln s_0 + b_0^+ = a_0^- \ln s_0 + b_0^-$, and its derivative discontinuous due to the Dirac delta in (9), $a_0^+/s_0 - a_0^-/s_0 = -1/s_0$. Summing explicitly the Fourier series (10) and requiring additionally that the Coulomb potential $v_{\text{Coul}}(s_1, s_2)$ be symmetric under exchange of 1 and 2 we find

$$v_{\text{Coul}}(s,\varphi;s_0,\varphi_0) = -\ln\frac{|z-z_0|}{h(s,s_0)} + a, \tag{12}$$

with $h(s, s_0) = 1$ or $h(s, s_0) = \sqrt{ss_0}$, and a a constant. Here if we imagine the plasma confined into a disk Ω_R of radius R we can choose

$$v_{\text{Coul}}(s,\varphi;s_0,\varphi_0) = -\ln\frac{|z-z_0|}{h(s,s_0)} + b, \tag{13}$$

with $h(s, s_0) = R$ and $b = a - \ln R$, or $h(s, s_0) = \sqrt{ss_0}$ and $b = a$, so that if we rescale all the s into λs and R into λR the Coulomb potential does not change apart from an additive constant. Imagine now we are on a plane [3], then $h(s, s_0) = R$. Then in the definition of the density (1) at any temperature we can change integration variables in the

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numerator from (s_i, φ_i) to $(x_i = s_i \mathrm{e}^{\mathrm{i}(\varphi_i-\varphi_1)}, y_i = \varphi_i - \varphi_1)$ for $i = 2, 3, \ldots, N$ with Jacobian 1. Calling $v_b = v_b(s/R) = \bar{\rho}\int_{\Omega_R} v_{\mathrm{Coul}}(s,\varphi;s',\varphi')\sqrt{g(s')}\,ds'\varphi'$ the neutralizing background potential and v_0 the self energy of the background we can write

$$\rho(s_1,\varphi_1) = \frac{N}{Z}\mathrm{e}^{-\beta[v_b(s_1/R)+v_0]}\int_{\Omega_R}\prod_{i>j\geq 2}\mathrm{e}^{-\beta v_{\mathrm{Coul}}(\mathbf{q}_i;\mathbf{q}_j)}\prod_{k=2}^{N}\left(\frac{|x_k - s_1|}{R}\right)^{\beta q^2} \times \mathrm{e}^{-\beta v_b\left(x_k \mathrm{e}^{-\mathrm{i}y_k}/R\right)}\sqrt{g\left(x_k \mathrm{e}^{-\mathrm{i}y_k}\right)}\,\mathrm{d}x_k\,\mathrm{d}y_k. \quad (14)$$

The integral does not depend on φ_1, so $\rho(s_1,\varphi_1) = \rho(s_1)$. Now we can make a change of variables where $s_k \to s_k/s_1$ for $k = 2, 3, \ldots, N$ and $R/s_1 \to T$ so that

$$\rho(s_1) = \frac{N}{Z}\mathrm{e}^{-\beta[v_b(1/T)+v_0]}\int_{\Omega_T}\prod_{i>j\geq 2}\mathrm{e}^{-\beta v_{\mathrm{Coul}}(\mathbf{q}_i;\mathbf{q}_j)}\prod_{k=2}^{N}\left(\frac{|x_k - 1|}{T}\right)^{\beta q^2} \times \mathrm{e}^{-\beta v_b(s_k/T)}\sqrt{g\left(s_k s_1\right)}\, s_1^{N-1}\,\mathrm{d}x_k\,\mathrm{d}y_k. \quad (15)$$

On a plane $\sqrt{g(ss_1)} = ss_1$, so that in equation (15) there is a multiplicative factor $s_1^{2(N-1)}$. So in the thermodynamic limit $T \to \infty$ and $N \to \infty$ we can say that $\rho(s_1) =$ constant, since we know that we must have a well defined thermodynamic limit. The same conclusion holds on a pseudosphere (see section 4.3.2 of [16]), on a cylinder (see equation (12a) of [9]), and on a Flamm's paraboloid (see section 4.2.4 of [18]). In these cases the explicit analytic expression of the density has been determined for the finite system as a function of the properties of the surface at the special value of the coupling constant $\beta q^2 = 2$. To the best of our knowledge there are no analytical results in the literature where the OCP has been found to have a non-constant number density in the thermodynamic limit on a given curved surface, and probably one has to resort to numerical simulations [25]. It certainly has to be expected that in a general curved surface the OCP in the thermodynamic limit may have a non-constant density, otherwise it would mean that an OCP in the plane has a uniform density for an arbitrary external field. It might actually be true that the effects of the metric and the background potential cancel one another when the potential is determined by Poisson's equation, but if it is true, it will be necessary to solve for the potential in more detail to prove it.

4. The quantum case

For the quantum fluid we find for the canonical ensemble distinguishable density matrix (the full density matrix for a system of bosons or fermions is then obtained by symmetrization or anti-symmetrization respectively) [26]

$$\rho_D(\mathbf{Q}',\mathbf{Q};\beta) = \int \rho_D(\mathbf{Q}',\mathbf{Q}((M-1)\tau);\tau)\cdots\rho_D(\mathbf{Q}(\tau),\mathbf{Q};\tau)\prod_{j=1}^{M-1}\sqrt{\tilde{g}_{(j)}}\prod_{\alpha=1}^{2N}\mathrm{d}Q^{\alpha}(j\tau), \quad (16)$$

where as usual we discretize the imaginary time in bits $\tau = \hbar\beta/M$ and $\mathbf{Q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ with

$$\tilde{g}_{(i)} = \det\|\tilde{g}_{\mu\nu}(\mathbf{Q}(i\tau))\|, \quad (17)$$

$$\tilde{g}_{\mu\nu}(\mathbf{Q}) = g_{\alpha_1\beta_1}(\mathbf{q}_1)\otimes\ldots\otimes g_{\alpha_N\beta_N}(\mathbf{q}_N), \quad (18)$$

to get to the path integral formulation, and in the small τ limit for particles of unitary mass it follows that

$$\rho(\mathbf{Q}(2\tau), \mathbf{Q}(\tau); \tau) = (2\pi\hbar)^{-N} \tilde{g}_{(2)}^{-1/4} \sqrt{D(\mathbf{Q}(2\tau), \mathbf{Q}(\tau); \tau)} \tilde{g}_{(1)}^{-1/4} \times \mathrm{e}^{\hbar\tau R(\mathbf{Q}(\tau))/12} \, \mathrm{e}^{-(1/\hbar) S(\mathbf{Q}(2\tau), \mathbf{Q}(\tau); \tau)}, \tag{19}$$

where R is the scalar curvature of the surface, S the action and D the van Vleck's determinant

$$D_{\mu\nu} = -\frac{\partial^2 S(\mathbf{Q}(2\tau), \mathbf{Q}(\tau); \tau)}{\partial Q^\mu(2\tau) \partial Q^\nu(\tau)}, \tag{20}$$

$$D(\mathbf{Q}(2\tau), \mathbf{Q}(\tau); \tau) = \det \|D_{\mu\nu}\|. \tag{21}$$

For example for free particles

$$\mathcal{H} = \tfrac{1}{2} \sum_{i=1}^{N} g^{\alpha_i \beta_i}(\mathbf{q}_i) p_{\alpha_i} p_{\beta_i} = \tfrac{1}{2} \sum_{i=1}^{N} g_{\alpha_i \beta_i}(\mathbf{q}_i) \dot{q}^{\alpha_i} \dot{q}^{\beta_i}, \tag{22}$$

$$S(\mathbf{Q}(2\tau), \mathbf{Q}(\tau); \tau) = K(\mathbf{Q}(2\tau), \mathbf{Q}(\tau); \tau) = \tfrac{1}{2} \sum_{i=1}^{N} d^2(\mathbf{q}_i(2\tau), \mathbf{q}_i(\tau))/\tau, \tag{23}$$

and for the fluid

$$S(\mathbf{Q}(2\tau), \mathbf{Q}(\tau); \tau) = K(\mathbf{Q}(2\tau), \mathbf{Q}(\tau); \tau) + \tau V(\mathbf{Q}(\tau)). \tag{24}$$

We then find the partition function through the integral

$$Z = \int \rho_D(\mathbf{Q}, \mathbf{Q}; \beta) \sqrt{\tilde{g}} \, \mathrm{d}\mathbf{Q}, \tag{25}$$

and the number density by

$$\sqrt{g(\mathbf{q}_1)} \rho(\mathbf{q}_1) = N \frac{\int \rho_D(\mathbf{Q}, \mathbf{Q}; \beta) \sqrt{\tilde{g}} \prod_{i=2}^{N} \mathrm{d}\mathbf{q}_i}{Z}. \tag{26}$$

It is then apparent that by choosing the same isometry on each imaginary time slice we reach the same conclusion as in section 3 for the classical (non-quantum) fluid.

5. Conclusions

We showed that in a surface of constant curvature without boundaries the local number density $\rho(\mathbf{q})$ of a non-ideal, $(V \neq 0)$, fluid is a constant in the thermodynamic limit. Clearly the ideal gas has constant density on every surface regardless of the curvature and of the thermodynamic limit.

The Coulomb potential for particles living on the surface depends on the metric tensor and is in general a function of the geodesic distance between the two charges. The *Coulomb fluid* density is a constant in the thermodynamic limit in the plane [3] the sphere [10] and the pseudosphere [15]–[17], all surfaces of constant curvature, but also on the Flamm's paraboloid [18], a surface of non-constant curvature.

We proposed a formulation for the number density which gives to the curvature of a surface with a conformal metric (the sphere, the pseudosphere and the Flamm's paraboloid

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are three surfaces of this kind) a physical interpretation as an additional external 'force' acting on the system of particles moving in the corresponding 'flat space'. The formulation, although suggestive, partly masks the intuition of the properties of the density because of the fact that the pair potential is inherently related to the properties of the curved surface, *i.e.* the geodesic distance between two points, which cannot be translated in terms of the properties of the corresponding fluid moving in the 'flat space' in a straightforward way. On the other hand the formulation suggests that the combination $\sqrt{g}\rho$ is a more fundamental quantity than just ρ itself. One can show both for the non-quantum and the quantum fluid that if ∂_{q^α} is a Killing vector field of the surface then if we can neglect surface effects $[\sqrt{g(\mathbf{q})}\rho(\mathbf{q})]_{,\alpha}=0$ and if $[\sqrt{g(\mathbf{q})}\rho(\mathbf{q})]_{,\alpha}=0,\ \forall\ v$ then $g_{,\alpha}=0$. These are the main results of our discussion. We can also say that $g_{,\alpha}=0$ *if and only if* $[\sqrt{g(\mathbf{q})}\rho(\mathbf{q})]_{,\alpha}=0,\ \forall\ v$.

The total potential energy of the fluid moving in the 'flat space' is $U(\mathbf{Q})=V(\mathbf{Q})+\sum_i\phi(\mathbf{q}_i;\beta)$, where the functional dependence on $\mathbf{Q}$ of the first term depends both on the fluid model, through $v(d(\mathbf{q}_i,\mathbf{q}_j))$, and the kind of surface, through d, whereas the functional form of the second term depends *only* on the kind of surface. It is then to be expected that given a fluid model the density can be non-constant on certain surfaces.

The OCP has uniform density on the cylinder (see equation (12a) of [9]), on the pseudosphere (see section 4.3.2 of [16]), and on the Flamm's paraboloid (see section 4.2.4 of [18]). In these cases the explicit expression of the density has been determined for the finite system as a function of the properties of the surface at the special value of the coupling constant $\beta q^2=2$. To the best of our knowledge there are no analytical results in the literature where the OCP has been found to have a non-constant number density in the thermodynamic limit on a given curved surface, and probably one has to resort to numerical simulations [25].

It would be important, in the future, to be able to understand if the surface effects on the finite system have some influence in the conclusion that if ∂_{q^α} is a Killing vector field of the surface then $[\sqrt{g(\mathbf{q})}\rho(\mathbf{q})]_{,\alpha}=0$ in the thermodynamic limit.

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References

[1] March N H and Tosi M P, 1984 *Coulomb Liquids* (New York: Academic Press)
[2] Edwards S F and Lenard A, 1962 *J. Math. Phys.* **3** 778
[3] Jancovici B, 1981 *Phys. Rev. Lett.* **46** 386
[4] Alastuey A and Jancovici B, 1981 *J. Physique* **42** 1
[5] Rosinberg M L and Blum L, 1984 *J. Chem. Phys.* **81** 3700
[6] Jancovici B, Manificat G and Pisani C, 1994 *J. Stat. Phys.* **76** 307
[7] Jancovici B and Téllez G, 1996 *J. Stat. Phys.* **82** 609
[8] Choquard Ph, 1981 *Helv. Phys. Acta* **54** 332
[9] Choquard Ph, Forrester P J and Smith E R, 1983 *J. Stat. Phys.* **33** 13
[10] Caillol J M, 1981 *J. Physique Lett.* **42** L–245
[11] Forrester P J, Jancovici B and Madore J, 1992 *J. Stat. Phys.* **69** 179
[12] Forrester P J and Jancovici B, 1996 *J. Stat. Phys.* **84** 337

[13] Téllez G and Forrester P J, 1999 *J. Stat. Phys.* **97** 489
[14] Jancovici B, 2000 *J. Stat. Phys.* **99** 1281
[15] Jancovici B and Téllez G, 1998 *J. Stat. Phys.* **91** 953
[16] Fantoni R, Jancovici B and Téllez G, 2003 *J. Stat. Phys.* **112** 27
[17] Jancovici B and Téllez G, 2004 *J. Stat. Phys.* **116** 205
[18] Fantoni R and Téllez G, 2008 *J. Stat. Phys.* **133** 449
[19] Fantoni R, 2012 *J. Stat. Mech.* P04015
[20] Tarjus G, Sausset F and Viot P, *Statistical mechanics of liquids and fluids in curved space*, 2011 *Advances in Chemical Physics* vol 148, ed S A Rice and A R Dinner (Hoboken, NJ: John Wiley) ch 4
[21] Hill T L, 1956 *Statistical Mechanics* (New York: McGraw-Hill)
[22] Misner C W, Thorne K S and Wheeler J A, 1973 *Gravitation* (New York: Freeman) Section 25.2
[23] Kühnel W, 2006 *Differential Geometry: Curves–Surfaces–Manifolds* 2nd edn (Providence, RI: American Mathematical Society) Liebmann theorem 4.51
[24] Rossi B, 1957 *Optics* (London: Addison-Wesley) Section 2–4
[25] Caillol J M, Levesque D, Weis J J and Hansen J P, 1982 *J. Stat. Phys.* **28** 325
[26] Schulman L S, 1981 *Techniques and Applications of Path Integrals* (New York: Wiley)