Two component plasma in a Flamm's paraboloid

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Abstract. The two component plasma (TCP) living in a Flamm's paraboloid is
studied at a value of the coupling constant $\Gamma = 2$ for which an analytic expression
for the grand canonical partition function is available. Two cases are considered:
the plasma in the half surface with an insulating horizon and the plasma in
the whole surface. The Green's function equation necessary to determine the
n-particle truncated correlation functions is explicitly found. In both cases
this proves too complicated to be solved analytically. Therefore we present a
method of solution reducing the problem to finding the two linearly independent
solutions of a linear homogeneous second order ordinary differential equation with
polynomial coefficients of high degrees. In the flat limit one recovers the solution
for the structure of the TCP in a plane in the first case but the collapse of opposite
charges at the horizon means that the structure of the plasma is physically not
well defined in the second case.

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Contents

1. Introduction

The two component plasma (TCP) is a neutral mixture of point wise particles of charge $\pm q$. The equation of state of the TCP living in a plane has been known since the work of Salzberg and Prager [1]. In the plasma the attraction between oppositely charged particles competes with the thermal motion and makes the partition function of the finite system diverge when $\Gamma = \beta q^2 \geq 2$, where $\beta = 1/k_{\mathrm{B}}T$ with k_{B} the Boltzmann constant. The system becomes unstable against the collapse of pairs of oppositely charged particles, and as a consequence all thermodynamic quantities diverge, so that the point particle model is well behaved only for $\Gamma < 2$ [2] when the Boltzmann factor for unlike particles is integrable at small separations of the charges. In this case rescaling the particle coordinates so as to stay in the unit disk one easily proves that the grand canonical partition function is a function of $\sqrt{\lambda_- \lambda_+} V^{(1-\Gamma/4)}$, where V is the volume of the plasma and $\lambda_\pm$ are the fugacities of the two charge species, and as a consequence the equation of state is $\beta P = n(1 - \Gamma/4)$, where $n = n_+ + n_-$ is the total particle number density. It also follows that the ratio $\sqrt{\lambda_- \lambda_+}/n^{(1-\Gamma/4)}$ must be a function of Γ only in the thermodynamic limit [3]. However, if the collapse is avoided by some short range repulsion (hard cores for instance), the model remains well defined for lower temperatures. Then, for $\Gamma > 4$ the long range Coulomb attraction binds positive and negative particles in pairs of finite polarizability. Thus, at

some critical value $\Gamma_c \sim 4$ the system undergoes the Kosterlitz–Thouless transition [4] between a high temperature ($\Gamma < 4$) conductive phase and a low temperature ($\Gamma > 4$) dielectric phase. The same behavior also occurs in the TCP living in one dimension [5].

The structure of the TCP living in a plane at the special value $\Gamma = 2$ of the coupling constant is solvable exactly analytically [6]. Through the use of an external potential it has also been studied in various confined geometries [7]–[10] and in a gravitational field [11, 12].

In curved surfaces it has only been studied in surfaces of constant curvature such as the sphere [13, 14] and the pseudosphere [15]. Unlike the one component plasma where the properties of the Vandermonde determinant allow analytical solution a Cauchy identity is used for the solution of the TCP. Unlike in the one component case where the solution is possible for the plasma confined in a region of the surface now this is not possible anymore without the use of an external potential. In these cases the external potential is rather given by $-(\Gamma/q^2) \ln \sqrt{g}$, where g is the determinant of the metric tensor.

In this work we study the TCP at the special value $\Gamma = 2$ of the coupling constant in a Flamm's paraboloid, a surface of non-constant curvature. In this surface the one component plasma [16] has been studied in various situations: confined to one 'universe' by an insulating or a grounded perfect conductor 'horizon' or in the whole surface. Here we will restrict ourselves to only the first and last situations.

In a curved surface, even though the finite system partition function will still be finite for $\Gamma < 2$ since the surface is locally flat, the structure will change with respect to the flat case. The purpose of the present work is to see how it changes for the special case $\Gamma = 2$ where an exact analytical solution is possible.

2. The Flamm's paraboloid $\mathcal{S}$

In this work, we want to study a two-dimensional TCP in the Flamm's paraboloid. This is a Riemannian surface $\mathcal{S}$ with the following metric:

$$\mathrm{d}\mathbf{s}^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2, \tag{1}$$

or $g_{rr} = 1/(1 - 2M/r), g_{\varphi\varphi} = r^2$, and $g_{r\varphi} = 0$.

The Flamm's paraboloid is an embeddable surface in the three-dimensional Euclidean space. It is composed of two identical 'universes': $\mathcal{S}_+$ and $\mathcal{S}_-$. These are both multiply connected surfaces connected by a hole of radius $2M$. We will from now on call the $r = 2M$ region of the surface its 'horizon'. The scalar curvature is $R = -2M/r^3$.

The system of coordinates (r, φ) with the metric (1) has the disadvantage that it requires two charts to cover the whole surface $\mathcal{S}$. It can be more convenient to use the variable

$$u = \pm\sqrt{\frac{r}{2M} - 1} \tag{2}$$

instead of r. This gives the following metric when using the system of coordinates (u, φ):

$$\mathrm{d}\mathbf{s}^2 = 4M^2(1 + u^2)[4\,\mathrm{d}u^2 + (1 + u^2)\,\mathrm{d}\varphi^2]. \tag{3}$$

The region $u > 0$ corresponds to $\mathcal{S}_+$ and the region $u < 0$ to $\mathcal{S}_-$.

The Laplace–Beltrami operator is

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\mu} \left(\sqrt{g}\, g^{\mu\nu} \frac{\partial}{\partial q^\nu} \right) f$$
$$= \left[\left(1 - \frac{2M}{r} \right) \partial_r^2 + \frac{1}{r^2} \partial_\varphi^2 + \left(\frac{1}{r} - \frac{M}{r^2} \right) \partial_r \right] f, \tag{4}$$

where $\mathbf{q} \equiv (r, \varphi)$. Finding the Green's function of the Laplacian, naturally [16] leads us to consider the system of coordinates (s, φ), with

$$s = (\sqrt{u^2 + 1} + u)^2. \tag{5}$$

The range for the variable s is $]0, +\infty[$. The lower paraboloid $\mathcal{S}_-$ corresponds to the region $0 < s < 1$ and the upper one $\mathcal{S}_+$ to the region $s > 1$. A point in the upper paraboloid with coordinate (s, φ) has a mirror image by reflection $(u \to -u)$ in the lower paraboloid, with coordinates $(1/s, \varphi)$, since if

$$s = (\sqrt{u^2 + 1} + u)^2 \tag{6}$$

then

$$\frac{1}{s} = (\sqrt{u^2 + 1} - u)^2. \tag{7}$$

In the upper paraboloid $\mathcal{S}_+$, the new coordinate s can be expressed in terms of the original one, r, as

$$s = \frac{(\sqrt{r} + \sqrt{r - 2M})^2}{2M}. \tag{8}$$

Using this system of coordinates, the metric takes the form of a flat metric multiplied by a conformal factor

$$\mathrm{d}\mathbf{s}^2 = \frac{M^2}{4} \left(1 + \frac{1}{s} \right)^4 (\mathrm{d}s^2 + s^2\, \mathrm{d}\varphi^2). \tag{9}$$

The Laplacian also takes a simple form

$$\Delta f = \frac{s}{\sqrt{g}} \Delta_{\mathrm{flat}} f, \tag{10}$$

where

$$\Delta_{\mathrm{flat}} f = \partial_s^2 f + \frac{1}{s} \partial_s f + \frac{1}{s^2} \partial_\varphi^2 f \tag{11}$$

is the Laplacian of the flat Euclidean space $\mathbb{R}^2$. The square root of the determinant of the metric is now given by $\sqrt{g} = (M/2)^2 s(1 + s^{-1})^4$.

It is useful to keep in mind the following small M behaviors:

$$s = 2\frac{r}{M} - 2 - \frac{1}{2}\frac{M}{r} - \frac{1}{2}\left(\frac{M}{r} \right)^2 - \frac{5}{8}\left(\frac{M}{r} \right)^3 + \mathrm{O}(M^4), \tag{12}$$

$$\sqrt{g} = \frac{rM}{2} + \frac{M^2}{2} + \frac{5M^3}{8r} + \mathrm{O}(M^4). \tag{13}$$

3. The Coulomb potential created by a point charge

We here summarize the results found in [16] on the determination of the Coulomb potential between two charges living in a half Flamm's paraboloid with an insulating horizon and between two charges living in the full Flamm's paraboloid.

The Coulomb potential $G(s, \varphi; s_0, \varphi_0)$ created at (s, φ) by a unit charge at (s_0, φ_0) is given by the Green's function of the Laplacian

$$\Delta G(s, \varphi; s_0, \varphi_0) = -2\pi\delta^{(2)}(s, \varphi; s_0, \varphi_0) \tag{14}$$

with appropriate boundary conditions. The Dirac distribution is given by

$$\delta^{(2)}(s, \varphi; s_0, \varphi_0) = \frac{1}{\sqrt{g}}\delta(s - s_0)\delta(\varphi - \varphi_0). \tag{15}$$

Notice that using the system of coordinates (s, φ) the Laplacian Green's function equation takes the simple form

$$\Delta_{\text{flat}}G(s, \varphi; s_0, \varphi_0) = -2\pi\frac{1}{s}\delta(s - s_0)\delta(\varphi - \varphi_0) \tag{16}$$

which is formally the same Laplacian Green's function equation as that for a flat space.

The Laplacian Green's function equation (14) can be solved, as usual, by using the decomposition as a Fourier series. Since equation (14) reduces to the flat Laplacian Green's function equation (16), the solution is the standard one

$$G(s, \varphi; s_0, \varphi_0) = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{s_<}{s_>}\right)^n \cos\left[n(\varphi - \varphi_0)\right] + g_0(s, s_0) \tag{17}$$

where $s_> = \max(s, s_0)$ and $s_< = \min(s, s_0)$.

We consider now the case of particles restricted to live in the half surface (hs) $\mathcal{S}_+$, $s > 1$, by a hard wall located at the 'horizon', $s = 1$. The region $\mathcal{S}_-$, $s < 1$, is empty and has the same dielectric constant as the upper region. We want to consider a model where the interaction potential reduces to the flat Coulomb potential in the flat limit $M \to 0$. The solution of the Laplacian Green's function equation is given in Fourier series by equation (17). The 0th order Fourier component g_0 can be determined by the requirement that, in the limit $M \to 0$, the solution reduces to the flat Coulomb potential

$$G^{\text{flat}}(\mathbf{r}, \mathbf{r}') = -\ln\frac{|\mathbf{r} - \mathbf{r}'|}{L}, \tag{18}$$

where L is an arbitrary constant length. Recalling that $s \sim 2r/M$, when $M \to 0$, we find

$$g_0(s, s_0) = -\ln s_> - \ln\frac{M}{2L} \tag{19}$$

and in the half surface

$$G^{\text{hs}}(s, \varphi; s_0, \varphi_0) = -\ln|z - z_0| - \ln\frac{M}{2L}, \tag{20}$$

where we defined $z = se^{i\varphi}$ and $z_0 = se^{i\varphi_0}$. And $\lim_{M \to 0} G^{\text{hs}} = G^{\text{flat}}$ as desired.

A similar procedure [16] gives the Coulomb potential between charges living in the whole surface (ws) $0 < s < \infty$,

$$G^{\text{ws}}(s, \varphi; s_0, \varphi_0) = -\ln\frac{|z - z_0|}{\sqrt{|zz_0|}} - \ln\frac{L_0}{L}, \tag{21}$$

with L_0 another length scale.

4. The two component plasma model

The TCP is a neutral mixture of two species of point charges of charge $\pm q$. In this work we want to study the plasma in the Flamm's paraboloid. In [16] the Coulomb potential between two charges living in the whole paraboloid, in a half surface with an insulating horizon, and in a half surface with a grounded horizon was found. The first and last cases have been summarized in section 3 as they will be the two cases considered next.

5. TCP in a half surface with an insulating horizon

When the TCP lives in a half surface with an insulating horizon the Coulomb potential is given by equation (20). We will use $u_i = s_i e^{i\varphi_i}$ and $v_j = s_j e^{i\varphi_j}$ to denote the complex coordinates of the positively and negatively charged particles respectively, where $s = (\sqrt{r} + \sqrt{r - 2M})^2/2M > 1$. Then, following [6], we use a Cauchy identity [17] to rewrite $e^{-\beta U_{2N}}$, where U_{2N} is the potential energy of a neutral system of N positive and N negative charges, as follows:

$$
\begin{aligned}
\exp\left\{ 2\sum_{i<j}\left[\ln\left|\frac{u_i - u_j}{2L/M}\right| + \ln\left|\frac{v_i - v_j}{2L/M}\right| \right] - 2\sum_{i,j}\ln\left|\frac{u_i - v_j}{2L/M}\right| \right\} \\
= \left(\frac{2L}{M}\right)^{2N}\left| \frac{\prod_{i<j}(u_i - u_j)(v_i - v_j)}{\prod_{i,j}(u_i - v_j)} \right|^2 \\
= \left(\frac{2L}{M}\right)^{2N}\left| \det\left(\frac{1}{u_i - v_j}\right)_{(i,j)\in\{1,...,N\}^2} \right|^2,
\end{aligned} \tag{22}
$$

where we had to choose the particular case of a coupling constant $\Gamma = \beta q^2 = 2$.

Following [6], to avoid divergences we start from a discretized model in which two sub-lattices U and V are introduced. The positive (negative) particles sit on the sub-lattice U (V). Each lattice site is occupied by no or one particle. The grand canonical partition function defined as a sum including only neutral systems is then

$$
\Xi = 1 + \sum_{N=1}^{\infty}\left(\frac{2L}{M}\right)^{2N}\lambda_-^N\lambda_+^N \sum_{\substack{u_1,...,u_N\in U \\ v_1,...,v_N\in V}}\left| \det\left(\frac{1}{u_i - v_j}\right)_{(i,j)\in\{1,...,N\}^2} \right|^2, \tag{23}
$$

where the sums are defined with the prescription that configurations which differ only by a permutation of identical particles are counted only once, and λ_p is the constant fugacity of the charges of species $p = \pm$. This grand partition function is the determinant of an anti-Hermitian matrix $\mathbf{M}$ explicitly shown in [7].

When passing to the continuum limit in the element $\mathbf{M}_{ij}$ one should replace u_i or v_i by z and u_j or v_j by z', i.e. $i \to z$ and $j \to z'$. Each lattice site is characterized by its complex coordinate z and an isospinor which is $\binom{1}{0}$ if the site belongs to the positive sub-lattice U and $\binom{0}{1}$ if it belongs to the negative sub-lattice V. We then define a matrix

$\mathcal{M}$ by

$$\langle z|\mathcal{M}|z'\rangle = \frac{\boldsymbol{\sigma_x} + \mathrm{i}\boldsymbol{\sigma_y}}{2}\frac{2L/M}{z - z'} + \frac{\boldsymbol{\sigma_x} - \mathrm{i}\boldsymbol{\sigma_y}}{2}\frac{2L/M}{\bar{z} - \bar{z}'}, \tag{24}$$

where the $\boldsymbol{\sigma}$s are the 2×2 Pauli matrices operating in the isospinor space, and $z = x + \mathrm{i}y$.

The matrix $\mathcal{M}$ can be expressed in terms of a simple Dirac operator

$$\langle z|\mathcal{M}|z'\rangle = \frac{2L}{M}(\boldsymbol{\sigma_x}\partial_x + \boldsymbol{\sigma_y}\partial_y)\ln|z - z'|, \tag{25}$$

and the grand partition function can be rewritten as

$$\begin{aligned}\Xi &= \det\left\{\mathbf{1}\delta^{(2)}(z;z') + \left[\lambda_+\frac{\mathbf{1}+\boldsymbol{\sigma_z}}{2} + \lambda_-\frac{\mathbf{1}-\boldsymbol{\sigma_z}}{2}\right]\langle z|\mathcal{M}|z'\rangle\right\} \\ &= \det[\mathbf{I} + \mathbf{K}^{-1}],\end{aligned} \tag{26}$$

with $\mathbf{I} = \mathbf{1}\delta^{(2)}(z;z')$ and

$$\boldsymbol{\lambda} = \lambda_+\frac{\mathbf{1}+\boldsymbol{\sigma_z}}{2} + \lambda_-\frac{\mathbf{1}-\boldsymbol{\sigma_z}}{2}, \tag{27}$$

$$\mathbf{K}^{-1} = \boldsymbol{\lambda}\mathcal{M}. \tag{28}$$

Then, since $\Delta_{\text{flat}}\ln|z| = 2\pi\delta(s)\delta(\varphi)/s = 2\pi\delta(z)$ (where $s = |z|$ and $\delta(z)$ is the flat Dirac delta function), the inverse operator $\mathbf{K}$ is

$$\mathbf{K} = \mathbf{m}^{-1}(z)\mathbf{O}, \tag{29}$$

where

$$\mathbf{m}(z) = m_+(z)\frac{\mathbf{1}+\boldsymbol{\sigma_z}}{2} + m_-(z)\frac{\mathbf{1}-\boldsymbol{\sigma_z}}{2}, \tag{30}$$

$$\mathbf{O} = \frac{2}{M}(\boldsymbol{\sigma_x}\partial_x + \boldsymbol{\sigma_y}\partial_y). \tag{31}$$

Here $m_\pm(z) = (2\pi L\lambda_\pm\sqrt{g}/sS)(2/M)^2$ are rescaled position dependent fugacities, $g = \det(g_{\mu\nu})$, and S is the area per lattice site which appears when the discrete sums are replaced by integrals. Notice that in the flat limit $M \to 0$ we find $m_\pm \to \tilde{m}_\pm$, where $\tilde{m}_\pm = 2\pi L\lambda_\pm/S$ are the rescaled fugacities of the flat system [7, 6]. Moreover S is a local property of the surface independent from its curvature.

We then find

$$\ln\Xi = \mathrm{Tr}\left\{\ln\left[\mathbf{I} + \mathbf{K}^{-1}\right]\right\},$$

and the one-body densities and n-body truncated densities can be obtained in the usual way by taking functional derivatives of the logarithm of the grand partition function with respect to the fugacities $\lambda_\pm$. Marking the sign of the particle charge at z_i by an index $p_i = \pm 1$, and defining the matrix

$$R_{p_1p_2}(z_1, z_2) = \langle z_1p_1|\mathbf{K}^{-1}(\mathbf{I} + \mathbf{K}^{-1})^{-1}|z_2p_2\rangle, \tag{32}$$

it can then be shown [6, 7] that they are given by

$$\rho_{p_1}^{(1)}(z_1) = R_{p_1 p_1}(z_1, z_1), \tag{33}$$

$$\rho_{p_1 p_2}^{(2)T}(z_1, z_2) = -R_{p_1 p_2}(z_1, z_2) R_{p_2 p_1}(z_2, z_1), \tag{34}$$

$$\rho_{p_1 p_2,\ldots,p_n}^{(n)T}(z_1, z_2, \ldots, z_n) = (-)^{n+1} \sum_{(i_1,i_2,\ldots,i_n)} R_{p_{i_1} p_{i_2}}(z_{i_1}, z_{i_2}) \cdots R_{p_{i_n} p_{i_1}}(z_{i_n}, z_{i_1}), \tag{35}$$

where $m_p(z) = (2\pi L \lambda_p \sqrt{g}/sS)(2/M)^2 = (\tilde{m}_p \sqrt{g}/s)(2/M)^2$, $\tilde{m}_p = 2\pi L \lambda_p/S$, $\sqrt{g} = (M/2)^2 s(1 + 1/s)^4$, and the summation runs over all cycles $(i_1, i_2, \ldots, i_n)$ built with $\{1, 2, \ldots, n\}$.

5.1. Symmetries of the Green's function R

Since $\mathbf{m}^\dagger(z) = \mathbf{m}(z)$ and $\mathbf{O}^\dagger = -\mathbf{O}$ we find

$$\overline{R_{p_1 p_2}(z_1, z_2)} = \langle z_2 p_2 | \mathbf{m}^{-1} \mathbf{O} (\mathbf{I} - \mathbf{m}^{-1} \mathbf{O})^{-1} (\mathbf{m}^{-1} \mathbf{O})^{-1} | z_1 p_1 \rangle. \tag{36}$$

Expanding in $\mathbf{O}$ (by using $(\mathbf{I} - \mathbf{m}^{-1}\mathbf{O})^{-1} = \mathbf{I} + \mathbf{m}^{-1}\mathbf{O} + (\mathbf{m}^{-1}\mathbf{O})^2 + \cdots$) and comparing with the definition $R_{p_1 p_2}(z_1, z_2) = \langle z_1 p_1 | (\mathbf{m}^{-1}\mathbf{O})^{-1}(\mathbf{I} + \mathbf{m}^{-1}\mathbf{O})^{-1} \mathbf{m}^{-1}\mathbf{O} | z_2 p_2 \rangle$ we find

$$\overline{R_{pp}(z_1, z_2)} = R_{pp}(z_2, z_1), \tag{37}$$

$$\overline{R_{p-p}(z_1, z_2)} = -R_{-pp}(z_2, z_1), \tag{38}$$

from which it also follows that $R_{pp}(z_1, z_1)$ has to be real. If $\lambda_+ = \lambda_-$ then we additionally must have

$$R_{pp}(z_1, z_2) = R_{-p-p}(z_1, z_2). \tag{39}$$

5.2. Two-body truncated correlation functions and the perfect screening sum rule

For the two-body truncated correlation functions of equation (34) we then find

$$\rho_{++}^{(2)T}(z_1, z_2) = -|R_{++}(z_1, z_2)|^2, \tag{40}$$

$$\rho_{+-}^{(2)T}(z_1, z_2) = |R_{+-}(z_1, z_2)|^2. \tag{41}$$

Notice that the total correlation function for the like particles $h_{++}(z_1, z_2) = \rho_{++}^{(2)T}(z_1, z_2)/\rho_+^{(1)}(z_1)\rho_+^{(1)}(z_2)$ goes to -1 when the particles coincide, $z_1 \to z_2$, as follows from the structure of equations (33) and (34). Moreover, the truncated densities of any order have to decay to zero as the two groups of particles are infinitely separated. In particular, $|R_{++}(z_1, z_2)| = |R_{++}(s1, s2; \varphi)|$, with $\varphi = \varphi_1 - \varphi_2$, has to decay to zero as $|s_1 - s_2| \to \infty$.

The perfect screening sum rule has to be satisfied for the symmetric mixture

$$\int [\rho_{+-}^{(2)T}(z_1, z_2) - \rho_{++}^{(2)T}(z_1, z_2)] \sqrt{g_1} \, ds_1 \, d\varphi_1 = \rho_\pm(z_2), \tag{42}$$

where g_1 is g calculated on particle 1.

5.3. From the structure to the thermodynamics

Following [12], we note that if we solve the eigenvalue problem $\mathbf{K}\boldsymbol{\psi}_n = \lambda_n \boldsymbol{\psi}_n$ (here $\boldsymbol{\psi}_n$ is a spinor of components $\psi_{+,n}$ and $\psi_{-,n}$), then

$$\Xi = \prod_n \left(1 + \frac{1}{\lambda_n}\right). \tag{43}$$

Now the eigenvalue problem reads

$$\{m_+(z) + A[m_-(z)]^{-1}A^\dagger\}\psi_-(z) = (m_-(z) - \lambda^2)\psi_-(z), \tag{44}$$

$$\{m_-(z) + A^\dagger[m_+(z)]^{-1}A\}\psi_+(z) = (m_-(z) - \lambda^2)\psi_+(z), \tag{45}$$

where $A = (2/M)(\partial_x + \mathrm{i}\partial_y)$ and $A^\dagger = (2/M)(-\partial_x + \mathrm{i}\partial_y)$ and the left-hand side is the same as the left-hand side for the Green's functions equations (48) and (49).

Of course we know that the thermodynamic quantities all diverge in our $\Gamma = 2$ case so equation (43), although suggestive, is of small practical interest.

5.4. Determination of the Green's function R

The Green's function matrix $\mathbf{R}$ is the solution of a system of four coupled partial differential equations, namely

$$(\mathbf{I} + \mathbf{K}^{-1})\mathbf{K}\mathbf{R}(z_1, z_2) = (\mathbf{I} + \mathbf{K})\mathbf{R}(z_1, z_2) = \mathbf{1}\delta^{(2)}(z_1; z_2), \tag{46}$$

where $\delta^{(2)}(z_1; z_2) = \delta^{(2)}(s, \varphi; s_0, \varphi_0) = (\sqrt{g})^{-1}\delta(s - s_0)\delta(\varphi - \varphi_0)$ is the Dirac delta function in the curved surface, $\delta(s - s_0)\delta(\varphi - \varphi_0)/s = \delta(s, \varphi; s_0, \varphi_0) = \delta(z; z_0)$ is the flat Dirac delta function, and $\mathbf{1}$ is the identity matrix. These can be rewritten as follows:

$$[\mathbf{O} + \mathbf{m}(z_1)]\mathbf{R}(z_1, z_2) = \mathbf{m}(z_1)\delta^{(2)}(z_1; z_2).$$

If instead of $\mathbf{R}$ one uses $\mathbf{R} = \mathbf{G}\tilde{\mathbf{m}}$, $\mathbf{G}$ satisfies the equation

$$[\mathbf{O} + \mathbf{m}(z_1)]\mathbf{G}(z_1, z_2) = \frac{4}{M^2}\mathbf{1}\delta(z_1; z_2), \tag{47}$$

where $\delta(z_1; z_2)$ is now the flat Dirac delta function[1].

By combining the components of this equation one obtains decoupled equations for G_{++} and G_{--} as follows:

$$\left\{m_+(z_1) + A^\dagger[m_-(z_1)]^{-1}A\right\}G_{++}(z_1, z_2) = \frac{4}{M^2}\delta(s_1, \varphi_1; s_2, \varphi_2), \tag{48}$$

$$\left\{m_-(z_1) + A[m_+(z_1)]^{-1}A^\dagger\right\}G_{--}(z_1, z_2) = \frac{4}{M^2}\delta(s_1, \varphi_1; s_2, \varphi_2), \tag{49}$$

while

$$G_{-+}(z_1, z_2) = -[m_-(z_1)]^{-1}AG_{++}(z_1, z_2), \tag{50}$$

$$G_{+-}(z_1, z_2) = +[m_+(z_1)]^{-1}A^\dagger G_{--}(z_1, z_2). \tag{51}$$

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[1] The same result could have been reached by noticing that the partition function of equation (23) rewritten in the continuum is equivalent to the partition function of the flat system with position dependent fugacities $\zeta_p(s) = \lambda_p(M/2)^2(1 + 1/s)^4(2L/M)$. This leads naturally [7] to definition of a Green's function $\tilde{\mathbf{G}} = (M/2)\mathbf{G}$ which satisfies the equation $[(M/2)\mathbf{O} + (2\pi/S)\boldsymbol{\zeta}(z)]\tilde{\mathbf{G}}(z, z') = \mathbf{1}\delta(z; z')$, which reduces to equation (47).

Then equation (48) can be rewritten in Cartesian coordinates as

$$\left\{ m_+(z_1)m_-(z_1) - \left(\frac{2}{M}\right)^2 \left[(\partial_{x_1}^2 + \partial_{y_1}^2) - \frac{4(-x_1 + \mathrm{i}y_1)}{s_1^2(1+s_1)}(\partial_{x_1} + \mathrm{i}\partial_{y_1})\right] \right\} G_{++}(z_1, z_2)$$

$$= \left(\frac{2}{M}\right)^4 \frac{\tilde{m}_- \sqrt{g_1}}{s_1^2}\delta(s_1 - s_2)\delta(\varphi_1 - \varphi_2) = \left(\frac{2}{M}\right)^4 \frac{\tilde{m}_- \sqrt{g_1}}{\sqrt{x_1^2 + y_1^2}}$$

$$\times\, \delta(x_1 - x_2)\delta(y_1 - y_2), \tag{52}$$

where $s = \sqrt{x^2 + y^2}$. From the expression of the gradient in polar coordinates it follows that

$$\partial_x = \cos\varphi\,\partial_s - \frac{\sin\varphi}{s}\partial_\varphi, \qquad \partial_y = \sin\varphi\,\partial_s + \frac{\cos\varphi}{s}\partial_\varphi, \tag{53}$$

Which allows us to rewrite equation (52) in polar coordinates as

$$\left[\tilde{m}_+\tilde{m}_- \left(1 + \frac{1}{s_1}\right)^8 - \left(\frac{2}{M}\right)^2 \left(\frac{1}{s_1}\partial_{s_1}(s_1\partial_{s_1}) + \frac{1}{s_1^2}\partial_{\varphi_1}^2 + \frac{4}{s_1(1+s_1)}\partial_{s_1} + \frac{4\mathrm{i}}{s_1^2(1+s_1)}\partial_{\varphi_1}\right)\right]$$

$$\times\, G_{++}(z_1, z_2) = \left(\frac{2}{M}\right)^4 \frac{\tilde{m}_- \sqrt{g_1}}{s_1^2}\delta(s_1 - s_2)\delta(\varphi_1 - \varphi_2). \tag{54}$$

From this equation we immediately see that $G_{++}(z_1, z_2)$ cannot be real. Notice that in the flat limit $M \to 0$ we have $s \sim 2r/M$ and equation (54) reduces to

$$\left[\tilde{m}_+\tilde{m}_- - \frac{1}{r_1}\partial_{r_1}(r_1\partial_{r_1}) - \frac{1}{r_1^2}\partial_{\varphi_1}^2\right] G_{++}(z_1, z_2) = \frac{\tilde{m}_-}{r_1}\delta(r_1 - r_2)\delta(\varphi_1 - \varphi_2), \tag{55}$$

which, when $\tilde{m}_+ = \tilde{m}_- = \tilde{m}$, has the following well known solution [7, 6]:

$$G_{++}(z_1, z_2) = \frac{\tilde{m}}{2\pi}K_0(\tilde{m}|\mathbf{r}_1 - \mathbf{r}_2|), \tag{56}$$

where K_0 is a modified Bessel function.

Let us from now on restrict ourselves to the case of equal fugacities of the two species. Then $\lambda_- = \lambda_+ = \lambda$ with

$$\tilde{m} = \frac{2\pi L}{S}\lambda = \frac{2\pi L e^{\beta\mu}}{\Lambda^2} = \left(2\pi L\frac{mq^2}{4\pi\hbar^2}\right)e^{2\mu/q^2}, \tag{57}$$

where $\hbar$ is Planck's constant, m is the mass of the particles, and μ is the chemical potential. So $\tilde{m}$ has the dimensions of an inverse length. From the symmetry of the problem we can say that $G_{++} = G_{++}(s_1, s_2; \varphi_1 - \varphi_2)$. We can then express the Green's function as the following Fourier series expansion:

$$G_{++}(s_1, s_2; \varphi) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} g_{++}(s_1, s_2; k)e^{\mathrm{i}k\varphi}. \tag{58}$$

Then, using the expansion of the Dirac delta function, $\sum_k e^{\mathrm{i}k\varphi} = 2\pi\delta(\varphi)$, we find that g_{++}, a continuous real function symmetric under exchange of s_1 and s_2, has to satisfy

the following equation:

$$[Q_0(k, s_1) + Q_1(s_1)\partial_{s_1} + Q_2(s_1)\partial_{s_1}^2]g_{++}(s_1, s_2; k) = \left(\frac{2}{M}\right)^2 \tilde{m} s_1^3 (1 + s_1)^5 \delta(s_1 - s_2), \quad (59)$$

where

$$Q_0(k, s) = \tilde{m}^2(1 + s)^9 + \left(\frac{2}{M}\right)^2 k s^6 (4 + k(1 + s)),$$

$$Q_1(s) = -\left(\frac{2}{M}\right)^2 s^7 (5 + s),$$

$$Q_2(s) = -\left(\frac{2}{M}\right)^2 s^8 (1 + s),$$

and the coefficients Q_i are polynomials of up to degree 9.

5.5. Method of solution

We start from the homogeneous form of equation (59). We note that, for a given k, the two linearly independent solutions $f_\alpha(s; k)$ and $f_\beta(s; k)$ of this linear homogeneous second order ordinary differential equation are not available in the mathematical literature to the best of our knowledge. Assuming that we knew these solutions we would then find the Green's function, $g_{++}(s_1, s_2; k)$, writing [18]

$$f(t_1, t_2; k) = c_k f_\alpha(s_<; k) f_\beta(s_>; k), \quad (60)$$

where $s_< = \min(s_1, s_2)$, $s_> = \max(s_1, s_2)$, and f_β has the correct behavior at large s. Then we determine c_k by imposing the kink in f due to the Dirac delta function at $s_1 = s_2$ as follows:

$$\partial_{s_1} f(s_1, s_2; k)|_{s_1 = s_2 + \epsilon} - \partial_{s_1} f(s_1, s_2; k)|_{s_1 = s_2 - \epsilon} = -\tilde{m}\frac{(1 + s_2)^4}{s_2^5}, \quad (61)$$

where ϵ is small and positive.

The Green's function, symmetric under exchange of s_1 and s_2, is reconstructed as follows:

$$G_{++}(z_1, z_2) = G_{++}(s_1, s_2; \varphi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c_k f_\alpha(s_<; k) f_\beta(s_>; k) e^{ik\varphi}. \quad (62)$$

6. TCP in the whole surface

In the whole surface, using equation (21), we can now write $e^{-\beta U_{2N}}$ at a coupling constant $\Gamma = 2$ as follows:

$$\left| \det\left(\frac{L}{L_0} \frac{\sqrt{|u_j v_j|}}{u_i - v_j} \right)_{(i,j)\in\{1,\ldots,N\}^2} \right|^2. \quad (63)$$

The grand partition function will then be

$$\Xi = \det\left[\mathbf{I} + \mathbf{K}_{\mathrm{ws}}^{-1}\right], \tag{64}$$

with

$$\mathbf{K}_{\mathrm{ws}}^{-1} = \left(\frac{M}{2L_0}\right)\boldsymbol{\lambda}_{\mathrm{ws}}\boldsymbol{\mathcal{M}}, \tag{65}$$

$$\boldsymbol{\lambda}_{\mathrm{ws}} = \lambda_+|z|\frac{\mathbf{1}+\boldsymbol{\sigma_z}}{2} + \lambda_-|z|\frac{\mathbf{1}-\boldsymbol{\sigma_z}}{2}. \tag{66}$$

The inverse operator is then given by

$$\mathbf{K}_{\mathrm{ws}} = \left(\frac{2L_0}{M}\right)\boldsymbol{\mathcal{M}}^{-1}\boldsymbol{\lambda}_{\mathrm{ws}}^{-1}, \tag{67}$$

$$\boldsymbol{\lambda}_{\mathrm{ws}}^{-1} = \frac{1}{\lambda_+|z|}\frac{\mathbf{1}+\boldsymbol{\sigma_z}}{2} + \frac{1}{\lambda_-|z|}\frac{\mathbf{1}-\boldsymbol{\sigma_z}}{2}, \tag{68}$$

$$\boldsymbol{\mathcal{M}}^{-1} = \frac{Ss}{2\pi L\sqrt{g}}\left(\frac{M}{2}\right)^2\mathbf{O}. \tag{69}$$

Introducing position dependent fugacities[2]

$$m_p(z) = \frac{2\pi(L/L_0)\lambda_p\sqrt{g}}{Ss} = \tilde{m}_p\frac{\sqrt{g}}{s}, \tag{70}$$

where now $\tilde{m}_p/L_0 \to \tilde{m}_p$, we can rewrite

$$\mathbf{K}_{\mathrm{ws}} = \frac{\boldsymbol{\sigma_x}+\mathrm{i}\boldsymbol{\sigma_y}}{2}a_- + \frac{\boldsymbol{\sigma_x}-\mathrm{i}\boldsymbol{\sigma_y}}{2}a_+, \tag{71}$$

with the operators

$$a_- = -\frac{\bar{z}}{m_-(z)|z|^3} + \frac{1}{m_-(z)|z|}(\partial_x - \mathrm{i}\partial_y), \tag{72}$$

$$a_+ = -\frac{z}{m_+(z)|z|^3} + \frac{1}{m_+(z)|z|}(\partial_x + \mathrm{i}\partial_y). \tag{73}$$

Then the equations for the Green's functions are

$$(1 - a_-a_+)R_{++}(z_1, z_2) = \delta^{(2)}(z_1; z_2), \tag{74}$$

$$(1 - a_+a_-)R_{--}(z_1, z_2) = \delta^{(2)}(z_1; z_2), \tag{75}$$

$$R_{+-} = -a_-R_{--}, \tag{76}$$

$$R_{-+} = -a_+R_{++}. \tag{77}$$

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[2] Alternatively we could have left the one-body terms ($\sqrt{|z|}$) outside the determinant, and then considered them as part of the position dependent fugacities, and simply applied the Cornu and Jancovici [7] formalism with these position dependent fugacities.

The equation for R_{++} in the symmetric mixture case is

$$\left[m^2(z_1) - \frac{2}{s_1^4} + \frac{2\partial_{s_1}}{s_1^3} - \frac{\partial_{s_1}^2}{s_1^2} - \frac{-\mathrm{i}\partial_{\varphi_1} + \partial_{\varphi_1}^2}{s_1^4} \right] R_{++}(z_1, z_2)$$

$$= \frac{m^2(z_1)}{\sqrt{g_1}} \delta(s_1 - s_2)\delta(\varphi_1 - \varphi_2) = \frac{\tilde{m}^2 \sqrt{g_1}}{s_1^2} \delta(s_1 - s_2)\delta(\varphi_1 - \varphi_2). \qquad (78)$$

From this equation we see that $R_{++}(z_1, z_2)$ will now be real.

By expanding equation (78) in a Fourier series in the azimuthal angle we now find

$$[Q_0(k, s_1) + Q_1(s_1)\partial_{s_1} + Q_2(s_1)\partial_{s_1}^2]g_{++}(s_1, s_2; k) = \left(\frac{M}{2} \right)^2 \tilde{m} s_1^3(1 + s_1)^4 \delta(s_1 - s_2), \qquad (79)$$

where

$$Q_0(k, s) = \left(\frac{M}{2} \right)^4 \tilde{m}^2(1 + s)^8 + s^4(k^2 - k - 2),$$

$$Q_1(s) = 2s^5,$$

$$Q_2(s) = -s^6,$$

and the coefficients Q_i are now polynomials of up to degree 8.

In the flat limit we find, for $G_{++} = R_{++}/\tilde{m}$, the following equation:

$$\left[\tilde{m}^2 - \frac{2}{r_1^4} + \frac{2\partial_{r_1}}{r_1^3} - \frac{\partial_{r_1}^2}{r_1^2} - \frac{-\mathrm{i}\partial_{\varphi_1} + \partial_{\varphi_1}^2}{r_1^4} \right] G_{++}(z_1, z_2) = \frac{\tilde{m}}{r_1} \delta(r_1 - r_2)\delta(\varphi_1 - \varphi_2). \qquad (80)$$

We then see that we now do not recover the TCP in the plane [7, 6]. This has to be expected because in the flat limit the Flamm's paraboloid reduces to two planes connected by the origin.

After the Fourier expansion of equation (58) we now get

$$[P_0(k, r_1) + P_1(r_1)\partial_{r_1} + P_2(r_1)\partial_{r_1}^2]g_{++}(r_1, r_2; k) = \tilde{m}\delta(r_1 - r_2), \qquad (81)$$

where

$$P_0(k, r) = \tilde{m}^2 r + \frac{k^2 - k - 2}{r^3},$$

$$P_1(r) = \frac{2}{r^2}, \qquad P_2(r) = -\frac{1}{r}.$$

The homogeneous form of this equation admits the following two linearly independent solutions:

$$\begin{aligned} f_1(r; -1) &= [D_{-1/2}(\mathrm{i}\sqrt{2\tilde{m}}r) + \overline{D_{-1/2}(\mathrm{i}\sqrt{2\tilde{m}}r)}]/2 \\ f_2(r; -1) &= D_{-1/2}(\sqrt{2\tilde{m}}r) \end{aligned} \qquad k = -1,$$

$$\begin{aligned} f_1(r; 2) &= [D_{-1/2}((-2)^{1/4}\sqrt{\tilde{m}}r) + \overline{D_{-1/2}((-2)^{1/4}\sqrt{\tilde{m}}r)}]/2 \\ f_2(r; 2) &= [D_{-1/2}(\mathrm{i}(-2)^{1/4}\sqrt{\tilde{m}}r) + \overline{D_{-1/2}(\mathrm{i}(-2)^{1/4}\sqrt{\tilde{m}}r)}]/2 \end{aligned} \qquad k = 2,$$

$$\begin{aligned} f_1(t; k) &= \sqrt{r}I_{-\sqrt{7-4k+4k^2}/4}(\tilde{m}r^2/2) \\ f_2(t; k) &= \sqrt{r}I_{\sqrt{7-4k+4k^2}/4}(\tilde{m}r^2/2) \end{aligned} \qquad \text{else,}$$

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where the $D_\nu(x)$ are parabolic cylinder functions and the $I_\mu(x)$ are modified Bessel functions of the first kind which diverge as $e^x/\sqrt{2\pi x}$ for large $x \gg |\mu^2 - 1/4|$.

Again we write $g_{++}(r_1, r_2; k) = c_k f_\alpha(r_<; k) f_\beta(r_>; k)$ and impose the kink condition,

$$\partial_{r_1} g_{++}(r_1, r_2; k)|_{r_1=r_2+\epsilon} - \partial_{r_1} g_{++}(r_1, r_2; k)|_{r_1=r_2-\epsilon} = -\tilde{m} r_2, \tag{82}$$

to find the c_k. The Green's function is then reconstructed using equation (62). But we immediately see that curiously $|G_{++}|$ diverges. Even the structure of the plasma is not well defined in this situation. The collapse of opposite charges at the horizon shrinking to the origin makes the structure of the plasma physically meaningless.

7. Conclusions

An attempt was made to extend the solution of the one component plasma, at the special value of the coupling constant $\Gamma = 2$, living in the Flamm's paraboloid [16] to the two component one, the TCP. The Flamm's paraboloid is a Riemannian surface with non-constant curvature which is asymptotically flat. Its curvature depends on a parameter M ($2M$ being the radius of the horizon) in such a way that for $M \to 0$ the surface becomes flat: two planes connected by the origin.

The work of Cornu and Jancovici [7, 6] on the TCP in a plane showed that at this particular value of the coupling constant, using a Cauchy identity, it is possible to find an analytic solution to the structure of the plasma starting from a discretized model to avoid divergences and taking the continuous limit in the end. All the n-particle correlation functions with $n > 1$ are well behaved except the particle density which diverges like all the other thermodynamic quantities due to the competition between the attraction of oppositely charged particles and the thermal motion. This can be prevented by adding a hard core to the particles in order to prevent collapse.

In this work we repeated the same calculation for the plasma living in the Flamm's paraboloid, in the half surface case, $s > 1$, with an insulating horizon (at $s = 1$) and in the full surface case, $s > 0$. The solution of the equation determining the structure, the Green's function equation, is reduced to the mathematical problem of finding the two linearly independent solutions of a linear homogeneous second order ordinary differential equation with polynomial coefficients of high degree (9th in the half surface case and 8th in the full surface case). To the best of our knowledge, an analytic solution of these equations turns out to be out of mathematical reach. Nonetheless the many-body problem has been reduced to finding the solutions of a simple differential equation. The further step of finding explicitly the analytical form of the solutions from the given differential equations is a matter of mathematical syntax and we think that it does not add much to the present work.

We discussed the symmetries of the Green's function, the perfect screening sum rule, the relationship between the structure and the thermodynamics, and the method of solution of the Green's function equation.

We found that for the plasma living in the half Flamm's paraboloid with an insulating horizon the flat limit $M \to 0$ reduces the system to the TCP on a plane [7, 6]. For the plasma living in the full Flamm's paraboloid taking the flat limit one does not recover the plasma in a plane and this has to be expected since the paraboloid in this limit reduces to two planes connected by a hole at the origin. Instead the resulting solution for the

structure turns out to be physically meaningless. The collapse of opposite charges at the horizon as it shrinks to the origin spoils the structure of the plasma.

We leave as an open problem the one of finding approximations to the two linearly independent solutions of the homogeneous counterpart of equation (59), for the half surface case, and of equation (79), for the whole surface case, as a function of the integer k, the further determination of the Green's function in these two cases, through equation (62), the eventual analysis of the solution and discussion about the necessity of the addition of a hard core to the particles (hard 'disks' of diameter D with $\delta s < 2D(1 + 1/s)^{-2}/M$) in order to cure the divergence of the densities ρ_p with the consequent determination of the equation of state. These final steps are necessary in order to have the problem of the structure of the fluid reduced to a closed form. In the present work we limited ourselves to what can be said in an analytical form.

Acknowledgment

References

[1] Salzberg A and Prager S, 1963 *J. Chem. Phys.* **38** 2587
[2] Hauge E H and Hemmer P C, 1971 *Phys. Nor.* **5** 209
[3] Šamaj L and Travěnec I, 2000 *J. Stat. Phys.* **101** 713
[4] Kosterlitz J M and Thouless D J, 1973 *J. Phys. C: Solid State Phys.* **6** 1181
[5] Lenard A, 1961 *J. Math. Phys.* **2** 682
[6] Cornu F and Jancovici B, 1987 *J. Stat. Phys.* **49** 33
[7] Cornu F and Jancovici B, 1989 *J. Chem. Phys.* **90** 2444
[8] Forrester P J, 1991 *J. Chem. Phys.* **95** 4545
[9] Téllez G and Merchán L, 2002 *J. Stat. Phys.* **108** 495
[10] Merchán L and Téllez G, 2004 *J. Stat. Phys.* **114** 735
[11] Téllez G, 1997 *J. Chem. Phys.* **106** 8572
[12] Téllez G, 1998 *J. Phys. A: Math. Gen.* **31** 5277
[13] Forrester P J, Jancovici B and Madore J, 1992 *J. Stat. Phys.* **69** 179
[14] Forrester P J and Jancovici B, 1996 *J. Stat. Phys.* **84** 337
[15] Jancovici B and Téllez G, 1998 *J. Stat. Phys.* **91** 953
[16] Fantoni R and Téllez G, 2008 *J. Stat. Phys.* **133** 449
[17] Aitken A C, 1956 *Determinants and Matrices* (New York: Interscience)
[18] Jackson J D, 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley) sections 3.9 and 3.11