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Pressures for a One-Component Plasma on a Pseudosphere

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The classical (i.e., non-quantum) equilibrium statistical mechanics of a two-dimensional one-component plasma (a system of charged point-particles embedded in a neutralizing background) living on a pseudosphere (an infinite surface of constant negative curvature) is considered. In the case of a flat space, it is known that, for a one-component plasma, there are several reasonable definitions of the pressure, and that some of them are not equivalent to each other. In the present paper, this problem is revisited in the case of a pseudosphere. General relations between the different pressures are given. At one special temperature, the model is exactly solvable in the grand canonical ensemble. The grand potential and the one-body density are calculated in a disk, and the thermodynamic limit is investigated. The general relations between the different pressures are checked on the solvable model.

KEY WORDS: Pseudosphere; negative curvature; two-dimensional one-component plasma; pressure; exactly solvable models.

1. INTRODUCTION

Coulomb systems such as plasmas or electrolytes are made of charged particles interacting through Coulomb's law. The simplest model of a Coulomb system is the one-component plasma (OCP), also called jellium: an assembly of identical point charges, embedded in a neutralizing uniform

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background of the opposite sign. Here we consider the classical (i.e., non-quantum) equilibrium statistical mechanics of the OCP. Although many features of more realistic systems are correctly reproduced, this model has the peculiarity that there are several reasonable definitions of its pressure, and some of these definitions are not equivalent to each other.[1,2]

The two-dimensional version (2D OCP) of the OCP has been much studied. Provided that the Coulomb potential due to a point-charge is defined as the solution of the Poisson equation in a two-dimensional world (i.e., is a logarithmic function $-\ln r$ of the distance r to that point-charge), the 2D OCP mimicks many generic properties of the three-dimensional Coulomb systems. Of course, this toy logarithmic model does not describe real charged particles, such as electrons, confined on a surface, which nevertheless interact through the three dimensional Coulomb potential $1/r$. One motivation for studying the 2D OCP is that its equilibrium statistical mechanics is exactly solvable at one special temperature: both the thermodynamical quantities and the correlation functions are available.[3]

How the properties of a system are affected by the curvature of the space in which the system lives is a question which arises in general relativity. This is an incentive for studying simple models. Thus, the problem of a 2D OCP on a pseudosphere has been considered.[4] A pseudosphere is a non-compact Riemannian surface of constant negative curvature. Unlike the sphere it has an infinite area and it is not embeddable in the three dimensional Euclidean space. The property of having an infinite area makes it interesting from the point of view of Statistical Physics because one can take the thermodynamic limit on it.

For this 2D OCP on a pseudosphere, the problem of studying and comparing the different possible definitions of the pressure also arises. This is the subject of the present paper. In Section 2, we give some basic properties of the pseudosphere and of a 2D OCP on it. In Section 3, we define the different pressures and derive general relations between them. In Section 4, we illustrate the general properties by considering the special temperature at which all properties can be explicitly and exactly calculated.

2. PSEUDOSPHERE AND ONE-COMPONENT PLASMA

2.1. The Pseudosphere

There are at least three commonly known sets of coordinates to describe a pseudosphere of Gaussian curvature $-1/a^2$. The one which renders explicit the resemblance with the sphere is $\vec{q}=(q^1,q^2)=(q^\tau,q^\varphi)=(\tau,\varphi)$ with $\tau\in[0,\infty[$ and $\varphi\in[0,2\pi[$, the metric being

$$ds^2 = g_{\alpha\beta}\,dq^\alpha\,dq^\beta = a^2(d\tau^2+\sinh^2\tau\,d\varphi^2) \tag{2.1}$$

Another set of coordinates often used is (r, φ) with $r/(2a) = \tanh(\tau/2)$. They are the polar coordinates of a disk of radius $2a$. The metric in terms of these new coordinates is

$$ds^2 = \frac{dr^2 + r^2\,d\varphi^2}{[1-(r^2/4a^2)]^2} \tag{2.2}$$

The disk with such a metric is called the Poincaré disk. Through an appropriate conformal transformation, the Poincaré disk can be mapped onto the upper half-plane with some metric, the Poincaré half-plane, but this latter representation will not be used here. The geodesic distance d_{01} between any two points $\vec{q}_0 = (\tau_0, \varphi_0)$ and $\vec{q}_1 = (\tau_1, \varphi_1)$ on the pseudosphere is given by

$$\cosh(d_{01}/a) = \cosh\tau_1 \cosh\tau_0 - \sinh\tau_1 \sinh\tau_0 \cos(\varphi_1 - \varphi_0) \tag{2.3}$$

Given the set of points at a geodesic distance from the origin less than or equal to d, that we shall call a disk of radius d, we can easily determine its circumference

$$\mathscr{C} = 2\pi a \sinh\left(\frac{d}{a}\right) \underset{d\to\infty}{\sim} \pi a e^{d/a} \tag{2.4}$$

and its area

$$\mathscr{A} = 4\pi a^2 \sinh^2\left(\frac{d}{2a}\right) \underset{d\to\infty}{\sim} \pi a^2 e^{d/a} \tag{2.5}$$

The Laplace–Beltrami operator on the pseudosphere is

$$\Delta = \frac{1}{a^2}\left(\frac{1}{\sinh\tau}\frac{\partial}{\partial\tau}\sinh\tau\,\frac{\partial}{\partial\tau} + \frac{1}{\sinh^2\tau}\frac{\partial^2}{\partial\varphi^2}\right) \tag{2.6}$$

2.2. The One-Component Plasma

The 2D OCP which is considered here is an ensemble of N identical point particles of charge q, constrained to move in a disk of radius $d = a\tau_0$ by an infinite potential barrier on the boundary of this domain $\tau = \tau_0$. The average particle number density is $n = N/\mathscr{A}$, where $\mathscr{A}$ is the area (2.5). There is a background with a charge density $\rho_b = -qn_b$ uniformly smeared on the disk (ρ_b is 0 outside the disk). It is convenient to introduce the number of elementary charges in the background: $N_b = n_b \mathscr{A}$. The total charge is not necessarily 0, thus in general $n_b \neq n$.

The pair Coulomb potential $v(d)$ between two unit charges, a geodesic distance d apart, satisfies the Poisson equation on the pseudosphere,

$$\Delta v(d) = -2\pi\delta^{(2)}(d) \tag{2.7}$$

where $\delta^{(2)}(d)$ is the Dirac delta function on the curved manifold. This Poisson equation admits a solution vanishing at infinity,

$$v(d) = -\ln\left[\tanh\left(\frac{d}{2a}\right)\right] \tag{2.8}$$

The electrostatic potential of the background $w(\vec{q})$ satisfies

$$\Delta w(\vec{q}) = -2\pi\rho_b \tag{2.9}$$

By symmetry, this electrostatic potential is only a function of τ. Expressing the Laplacian (2.6) in terms of the variable $\cosh\tau$, and requesting the solution to be regular at $\tau = 0$ and to have the correct value at $\tau = \tau_0$ (corresponding to the background total charge), one finds the solution

$$w(\tau) = 2\pi a^2 q n_b \left\{\ln\left[\frac{1-\tanh^2(\tau_0/2)}{1-\tanh^2(\tau/2)}\right] + \sinh^2(\tau_0/2)\ln[\tanh^2(\tau_0/2)]\right\} \tag{2.10}$$

Let $dS = 2\pi a^2 \sinh\tau\, d\tau$ be an area element. The self energy of the background is

$$\begin{aligned} v_0 &= \tfrac{1}{2}\int_{\tau<\tau_0} \rho_b w(\tau)\, dS \\ &= (2\pi a^2 q n_b)^2 \{\sinh^2(\tau_0/2) - \ln[\cosh^2(\tau_0/2)] \\ &\quad - \sinh^4(\tau_0/2)\ln[\tanh^2(\tau_0/2)]\} \end{aligned} \tag{2.11}$$

The total potential energy of the system is

$$U = v_0 + v_{pb} + v_{pp} \tag{2.12}$$

where v_{pp} is the potential energy due to the interactions between the particles,

$$v_{pp} = \tfrac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N} q^2 v(d_{ij}) \tag{2.13}$$

and v_{pb} is the potential energy due to the interaction between the particles and the background,

$$v_{pb} = \sum_{i=1}^{N} q w(\tau_i) \tag{2.14}$$

3. THE DIFFERENT PRESSURES AND THEIR RELATIONS

In the case of a flat system, the pressure which is often considered, termed the thermal pressure, is defined from the free energy F by the standard relation $P^{(\theta)} = -(\partial F / \partial \mathscr{A})_{\beta, N, N_b}$, where β is the inverse temperature. In the case of a flat neutral ($N = N_b$) 2D OCP, this thermal pressure is given by the simple exact expression $\beta P^{(\theta)} = n[1-(\beta q^2/4)]$.[5,6] Thus, this thermal pressure becomes negative for $\beta q^2 > 4$, i.e., at low temperatures. This pathology of the OCP occurs also in three dimensions; it is related to the presence of an inert background without kinetic energy. Indeed, the uniform background can be considered as the limit of a gas of negative particles of charge $-\epsilon$ and number density ν, when $\epsilon \to 0$, $\nu \to \infty$, for a fixed value of the charge density $-\epsilon\nu$. In this limit, the ideal-gas part (kinetic part) of the background average energy density becomes infinite. In the OCP Hamiltonian, this infinite energy density is omitted. The price paid for this omission is that the corresponding (infinite) ideal-gas contribution to the pressure is omitted, and the remaining pressure may be negative.[4]

Unhappy with this negativeness, Choquard *et al.*[1] and Navet *et al.*[2] have introduced another pressure, the kinetic pressure $P^{(k)}$, which is the pressure exerted on the wall by the particles of charge q only. This kinetic pressure turns out to be also the one which is obtained through the use of the virial theorem. Although for usual fluids the thermal and kinetic pressures are equivalent, in the presence of a background they are different, with the kinetic pressure being always positive. This positiveness led the above authors to argue that the kinetic pressure is the "right" one. Anyhow, a detailed comparison of the diverse possible definitions of the pressure of a flat OCP has been done.[1]

In the present paper, it is this comparison that we extend to the case of a 2D OCP on a pseudosphere. We shall restrict ourselves to the case of a domain in the shape of a disk. We are especially interested in the thermodynamic limit, i.e., when the disk radius becomes infinite, for fixed values of β, n, n_b.

[4] In the case of a two-dimensional *two*-component plasma made of point-particles, the pressure also becomes negative when extrapolated to low temperatures $\beta q^2 > 4$. However, now $\beta q^2 > 4$ is outside the domain of definition of the partition function.

3.1. Kinetic and Virial Pressures

The average force exerted by the particles on a perimeter element ds is $(1/\beta)\, n^{(1)}(\tau_0)\, ds$, where $n^{(1)}(\tau)$ is the one-body density at the distance $a\tau$ from the origin. Therefore, the kinetic pressure is

$$P^{(k)} = (1/\beta)\, n^{(1)}(\tau_0) \tag{3.1}$$

We assume that this quantity has a limit when $\tau_0 \to \infty$. In Section 4, this assumption will be checked in the special case $\beta q^2 = 2$. It will now be shown that the virial pressure $P^{(v)}$, i.e., the pressure computed from the virial theorem, is the same as $P^{(k)}$.

In terms of the $2N$ coordinate components q^N and $2N$ conjugate momentum components p^N, the Hamiltonian of our OCP of N particles is

$$H(q^N, p^N) = T(q^N, p^N) + \bar{U}(q^N) \tag{3.2}$$

where $\bar{U} = U +$ confining potential and the kinetic energy T is

$$T = \frac{1}{2m} \sum_{i=1}^{N} g^{\alpha\beta}(\vec{q}_i)\, p_{i\alpha}\, p_{i\beta} \tag{3.3}$$

The Roman indices label the particles, and the lower or upper Greek indices denote covariant or contravariant components, respectively. As usual, a sum over repeated Greek indices is tacitly assumed. The equations of motion for particle i are

$$\begin{cases} \dot{q}_i^{\alpha} = \dfrac{\partial H}{\partial p_{i\alpha}} = \dfrac{1}{m}\, g^{\alpha\beta}(\vec{q}_i)\, p_{i\beta} \\[2ex] \dot{p}_{i\alpha} = -\dfrac{\partial H}{\partial q_i^{\alpha}} = -\dfrac{1}{2m}\, \dfrac{\partial g^{\beta\gamma}}{\partial q_i^{\alpha}}\, p_{i\beta}\, p_{i\gamma} - \dfrac{\partial \bar{U}}{\partial q_i^{\alpha}} \end{cases} \tag{3.4}$$

where the dot stands for total derivative with respect to time. If we take the time derivative of $\sum_i q_i^{\tau} p_{i\tau} = \sum_i \tau_i\, p_{i\tau}$, we find[5]

$$\frac{d}{dt} \sum_i \tau_i\, p_{i\tau} = \frac{1}{m} \sum_{i=1}^{N} g^{\tau\beta}(\vec{q}_i)\, p_{i\tau}\, p_{i\beta} - \frac{1}{2m} \sum_{i=1}^{N} \tau_i\, \frac{\partial g^{\beta\gamma}}{\partial \tau_i}\, p_{i\beta}\, p_{i\gamma} - \sum_{i=1}^{N} \tau_i\, \frac{\partial \bar{U}}{\partial \tau_i} \tag{3.5}$$

[5] One may be tempted to start with the time derivative of $\sum_i q_i^{\alpha} p_{i\alpha} = \sum_i (\tau_i\, p_{i\tau} + \varphi_i\, p_{i\varphi})$. Note however that this quantity does not remain finite at all times. This is because, when one follows the motion of a particle colliding with the boundary, it may go around the origin indefinitely, and φ_i (which must be defined as a continuous variable, without any 2π jumps) may increase indefinitely. Thus the time average of the time derivative of this quantity does not vanish.

where the last term is called the virial of the system. Since the system is confined in a finite domain, the coordinates $\tau_i(t)$ and their canonically conjugated momenta $p_{i\tau}(t)$ remain finite at all times. Thus,

$$\left\langle \frac{d}{dt} \sum_{i=1}^{N} \tau_i \, p_{i\tau} \right\rangle_t = 0 \tag{3.6}$$

where $\langle \cdots \rangle_t$ denotes a time average. Assuming that the system is ergodic, we can replace time averages by microcanonical averages. Assuming the equivalence of ensembles in the thermodynamic limit, we can as well use canonical or grand-canonical averages $\langle \cdots \rangle$. In the present section, we use canonical averages. The average of the r.h.s. of (3.5) vanishes. Separating in the last term of (3.5) the contribution from the forces exerted by the walls, which is, in the average, $-a\tau_0 \mathscr{C} P^{(v)}$, we obtain

$$a\tau_0 \mathscr{C} P^{(v)} = \left\langle \frac{1}{m} \sum_{i=1}^{N} g^{\tau\beta}(\vec{q}_i)\, p_{i\tau}\, p_{i\beta} \right\rangle - \left\langle \frac{1}{2m} \sum_{i=1}^{N} \tau_i \, \frac{\partial g^{\beta\gamma}}{\partial \tau_i}\, p_{i\beta}\, p_{i\gamma} \right\rangle - \left\langle \sum_{i=1}^{N} \tau_i \, \frac{\partial U}{\partial \tau_i} \right\rangle \tag{3.7}$$

We now calculate the three terms in the r.h.s. of (3.7). The first one is the average of twice a contribution to the Hamiltonian, which is quadratic in the N variables $p_{i\tau}$ (g is diagonal); since the average of a quadratic term in the Hamiltonian is $1/(2\beta)$, the first term in the r.h.s. of (3.7) is

$$\left\langle \frac{1}{m} \sum_{i=1}^{N} g^{\tau\tau}(\vec{q}_i)(p_{i\tau})^2 \right\rangle = \frac{N}{\beta} \tag{3.8}$$

The second term reduces to $-\langle (1/2m) \sum_{i=1}^{N} \tau_i (\partial g^{\varphi\varphi}/\partial \tau_i)(p_{i\varphi})^2 \rangle$. Averaging first on $p_{i\varphi}$ replaces $(p_{i\varphi})^2/2m$ by $1/[2\beta g^{\varphi\varphi}(\tau_i)]$. The second term becomes

$$\frac{1}{\beta} \left\langle \sum_{i=1}^{N} \frac{\tau_i}{\tanh \tau_i} \right\rangle = \frac{1}{\beta} \int_{\tau < \tau_0} n^{(1)}(\tau) \, \frac{\tau}{\tanh \tau} \, dS \tag{3.9}$$

Finally, since

$$\frac{dn^{(1)}(\tau_1)}{d\tau_1} = -\beta N \, \frac{\int e^{-\beta U} (\partial U/\partial \tau_1) \, dS_2 \cdots dS_N}{\int e^{-\beta U} \, dS_1 \, dS_2 \cdots dS_N} \tag{3.10}$$

the third term can be written as

$$-N \left\langle \tau_1 \, \frac{\partial U}{\partial \tau_1} \right\rangle = \frac{1}{\beta} \int_{\tau_1 < \tau_0} \tau_1 \, \frac{dn^{(1)}(\tau_1)}{d\tau_1} \, dS_1 \tag{3.11}$$

Putting together the contributions (3.8), (3.9), and (3.11) gives for (3.7)

$$a\tau_0 \mathscr{C} P^{(v)} = \frac{N}{\beta} + \frac{1}{\beta}\int_0^{\tau_0} \left[n^{(1)}(\tau)\frac{\tau}{\tanh\tau} + \tau\frac{dn^{(1)}(\tau)}{d\tau} \right] 2\pi a^2 \sinh\tau \, d\tau \tag{3.12}$$

After an integration by parts, (3.12) becomes

$$P^{(v)} = \frac{1}{\beta} n^{(1)}(\tau_0) = P^{(k)} \tag{3.13}$$

3.2. The Thermal Pressure

The thermal pressure is defined as

$$P^{(\theta)} = -\left(\frac{\partial F}{\partial \mathscr{A}}\right)_{\beta, N, N_b} \tag{3.14}$$

where F is the free energy. This expression is appropriate for the canonical ensemble, since F is related to the canonical partition function Z by $\beta F = -\ln Z$.

3.2.1. The Thermal Pressure in the Grand Canonical Ensemble

In the following, we shall also need an expression of the thermal pressure appropriate for the grand canonical ensemble. It should be remembered that, for a flat OCP in three dimensions, the grand canonical partition function must be defined[7] as an ensemble of systems with any number N of particles in a fixed volume and *with a fixed background charge density* $-qn_b$ (using an ensemble of neutral systems, i.e., varying n_b together with N would give a divergent grand partition function). In two dimensions, β times the free energy for a neutral flat system[3] behaves as $[1-(\beta q^2/4)]\, N \ln N$ as $N \to \infty$, and therefore the neutral grand canonical partition function diverges if $\beta q^2 > 4$. This indicates that, in the present case of a 2D OCP on a pseudosphere, a similar divergence might occur for an ensemble of neutral systems, and we prefer to use an ensemble with a fixed background (which, furthermore, will be seen to be exactly solvable at $\beta q^2 = 2$). Thus, the grand partition function Ξ and the corresponding grand potential $\Omega = -(1/\beta)\ln \Xi$ are functions of β, $\mathscr{A}$, ζ, n_b, where ζ is the fugacity. The usual Legendre transformation from F to Ω and from N to ζ changes (3.14) into

$$P^{(\theta)} = -\left(\frac{\partial \Omega}{\partial \mathscr{A}}\right)_{\beta, \zeta, N_b} \tag{3.15}$$

We assume that, even on a pseudosphere, the grand potential is extensive, i.e., of the form $\Omega = \mathscr{A}\omega(\beta, \zeta, n_b)$. Since ω depends on $\mathscr{A}$ through $n_b = N_b/\mathscr{A}$, Eq. (3.15) becomes

$$P^{(\theta)} = -\omega + n_b \frac{\partial \omega}{\partial n_b} \tag{3.16}$$

Note the difference with an ordinary fluid, without a background, for which $P^{(\theta)} = -\omega$.

3.2.2. The $P^{(\theta)} - P^{(k)}$ Difference

For a OCP, the thermal pressure is different from the kinetic pressure. In the case of a 2D OCP in a flat disk, in the thermodynamic limit, the boundary becomes a straight line and the difference was found to be[1]

$$P^{(\theta)} - P^{(k)} = -2\pi q^2 n_b \int_0^\infty [n^{(1)}(x) - n_b]\, x\, dx \tag{3.17}$$

where $n^{(1)}(x)$ is the density at distance x from the boundary. Using the Poisson equation, one can write (3.17) in the equivalent form[8]

$$P^{(\theta)} - P^{(k)} = q n_b [\phi_{\text{surface}} - \phi_{\text{bulk}}] \tag{3.18}$$

where ϕ_{bulk} and ϕ_{surface} are the electric potential in the bulk and on the disk boundary, respectively.[6]

Equation (3.18) can be proven as follows. Either in the flat case, or in the case of a pseudosphere, let us consider a large disk of area $\mathscr{A}$, filled with a 2D OCP. For compressing it infinitesimally, changing the area by $d\mathscr{A} < 0$, at constant β, N, N_b, we must provide the reversible work $\delta W = -P^{(\theta)}\, d\mathscr{A}$. We may achieve that compression in two steps. First, one compresses the particles only, leaving the background behind; the corresponding work is $\delta W^{(1)} = -P^{(k)}\, d\mathscr{A}$, since $P^{(k)}$ is the force per unit length exerted on the wall by the particles alone. Then, one compresses the background, i.e., brings the charge $q n_b\, d\mathscr{A}$ from a region where the potential is ϕ_{surface} into the plasma where the potential is $\phi(r)$, spreading it uniformly; the corresponding work is $\delta W^{(2)} = q n_b\, d\mathscr{A}[(1/\mathscr{A}) \int \phi(r)\, dS - \phi_{\text{surface}}]$, where $\phi(r)$ is the potential at distance r from the center. Therefore,

$$P^{(\theta)} - P^{(k)} = q n_b \left[\phi_{\text{surface}} - \frac{1}{\mathscr{A}} \int \phi(r)\, dS \right] \tag{3.19}$$

[6] In the original papers,[1, 8] the derivations of (3.17) and (3.18) have been done in the case of a neutral system. However, these derivations can be easily extended to systems carrying a total non vanishing charge.

We expect $\phi(r)$ to differ from ϕ_{bulk} only in the neighborhood of the boundary circle.

In the case of a flat disk, the contribution of this neighborhood to the integral in (3.19) is negligible in the thermodynamic limit, $\phi(r)$ can be replaced by the constant ϕ_{bulk}, and one obtains (3.18). On a pseudosphere, (3.19) [with $\phi(\tau)$ instead of $\phi(r)$] is still valid. However, now, in the large-disk limit, the integration element $dS = 2\pi a^2 \sinh \tau \, d\tau$ makes the boundary neighborhood dominant, and we rather have

$$P^{(\theta)} - P^{(k)} \sim q n_b \left[\phi(\tau_0) - e^{-\tau_0} \int_0^{\tau_0} \phi(\tau) \, e^{\tau} \, d\tau \right] \tag{3.20}$$

After some manipulations, in the thermodynamic limit, (3.20) can be shown to be equivalent to

$$P^{(\theta)} - P^{(k)} = -2\pi a^2 n_b q^2 \int_0^{\infty} [n^{(1)}(\sigma) - n_b] \, \sigma e^{-\sigma} \, d\sigma \tag{3.21}$$

where we have introduced the variable $\sigma = \tau_0 - \tau$ and $n^{(1)}(\sigma)$ now denotes the particle density at distance $a\sigma$ from the boundary. Indeed, in (3.21), $n^{(1)}(\tau) - n_b$ can be expressed in terms of $\phi(\tau)$ through the Poisson equation $\Delta\phi(\tau) = -2\pi q[n^{(1)}(\tau) - n_b]$. Since the charge density is localized at large τ, we can use for the Laplacian $\Delta \sim a^{-2}[d^2/d\tau^2 + d/d\tau]$. After integrations by parts, (3.20) is recovered.

In conclusion, (3.17) valid for a large flat disc generalizes into (3.21) on a pseudosphere. In the limit $a \to \infty$, $\sigma \to 0$, $a\sigma = x$, Eq. (3.21) does reproduce (3.17).

3.3. The Mechanical Pressure

Choquard *et al.*[1] have also defined a mechanical pressure, in terms of the free energy F, as

$$P^{(m)} = -\left(\frac{\partial F}{\partial \mathscr{A}} \right)_{\beta, N, n_b} \tag{3.22}$$

In terms of the grand potential Ω, a Legendre transformation now gives

$$P^{(m)} = -\left(\frac{\partial \Omega}{\partial \mathscr{A}} \right)_{\beta, \zeta, n_b} \tag{3.23}$$

If the grand potential is extensive, of the form $\Omega = \mathscr{A}\omega(\beta, \zeta, n_b)$, (3.23) gives

$$P^{(m)} = -\omega \tag{3.24}$$

The difference $P^{(m)} - P^{(k)}$ can be obtained by a slight change in the argument of Section 3.2.2. Again, we consider a large disk filled with a 2D OCP of area $\mathscr{A}$, and we compress it infinitesimally, changing its area by $d\mathscr{A} < 0$, now at constant β, N, n_b, providing the reversible work $\delta W = -P^{(m)}\, d\mathscr{A}$, in two steps. Again, first one compresses the particles only, leaving the background behind, and the corresponding work is $\delta W^{(1)} = -P^{(k)}\, d\mathscr{A}$. Then, one must withdraw the leftover background charge $qn_b\, d\mathscr{A}$, bringing it from the surface where the potential is ϕ_{surface} to infinity where the potential vanishes. The corresponding work is $\delta W^{(2)} = -qn_b\, d\mathscr{A}\, \phi_{\text{surface}}$. Therefore, for a disk on a pseudosphere, $P^{(m)} - P^{(k)} = qn_b\phi_{\text{surface}}$.[7]

In the thermodynamic limit, $\phi_{\text{surface}} \to 2\pi a^2 q(n - n_b)$ and

$$P^{(m)} - P^{(k)} = 2\pi a^2 q^2 n_b (n - n_b) \tag{3.25}$$

This difference vanishes for a neutral system ($n = n_b$).

The relations (3.21) and (3.25) between the different pressures obtained here by means of electrostatic arguments can also be obtained in a more formal way following Choquard *et al.*,[(1)] using the dilatation method (doing a change of variable $\tau = \tau_0 \tilde{\tau}$ in the partition function to explicitly show the area $\mathscr{A}$ dependence) and the BGY equations to replace the two-body density terms that appear in the calculations by one-body density terms.

3.4. The Maxwell Tensor Pressure

On a pseudosphere, since the area of a large domain is of the same order as the area of the neighborhood of the boundary, all the above definitions of the pressure depend on the boundary conditions. In previous papers, a definition of a bulk pressure independent of the boundary conditions has been looked for. After an erroneous attempt,[(4)] it has been

[7] This result is identical with the one obtained by Choquard *et al.*[(1)] in the case of a flat disk. However, their general formula might make difficulties in two dimensions, because the Coulomb potential $-\ln(r/L)$ does not vanish at infinity and involves an arbitrary constant length L. These difficulties do not arise on a pseudosphere.

argued[9, 10] that a bulk pressure P_{Maxwell} could be defined from the Maxwell stress tensor[11] at some point well inside the fluid. The result was

$$\beta P_{\text{Maxwell}} = n_b \left(1 - \frac{\beta q^2}{4}\right) \tag{3.26}$$

That same equation of state holds for the 2D OCP on a plane, a sphere, or a pseudosphere.

4. EXACT RESULTS AT $\beta q^2 = 2$

When the Coulombic coupling constant is $\beta q^2 = 2$, all the thermodynamic properties and correlation functions of the two-dimensional one-component plasma can be computed exactly in several geometries[3, 12, 13] including the pseudosphere.[4] In ref. 4 the density and correlation functions in the bulk, on a pseudosphere, were computed. Here we are interested in these quantities near the boundary. In ref. 4 the calculations were done for a system with a $-\ln \sinh(d/2a)$ interaction and it was shown that this interaction gives the same results as the real Coulomb interaction $-\ln \tanh(d/2a)$, as far as the bulk properties are concerned. The argument in favor of this equivalence no longer holds for the density and other quantities near the boundary; therefore we shall concentrate on the real Coulomb system with a $-\ln \tanh(d/2a)$ interaction. This system was briefly considered in the Appendix of ref. 4. For the sake of completeness, we revisit here the reduction of the statistical mechanics problem to the study of a certain operator.

4.1. The Grand Potential

Working with the set of coordinates (r, φ) on the pseudosphere (the Poincaré disk representation), the particle i-particle j interaction term in the Hamiltonian can be written as[4]

$$v(d_{ij}) = -\ln \tanh(d_{ij}/2a) = -\ln \left| \frac{(z_i - z_j)/(2a)}{1 - (z_i \bar{z}_j / 4a^2)} \right| \tag{4.1}$$

where $z_j = r_j e^{i\varphi_j}$ and $\bar{z}_j$ is the complex conjugate of z_j. This interaction (4.1) happens to be the Coulomb interaction in a flat disc of radius $2a$ with ideal conductor walls. Therefore, it is possible to use the techniques which have been developed[14, 15] for dealing with ideal conductor walls, in the grand canonical ensemble.

The grand canonical partition function of the OCP at fugacity ζ with a fixed background density n_b, when $\beta q^2 = 2$, is

$$\Xi = C_0 \left[1 + \sum_{N=1}^{\infty} \frac{1}{N!} \int \prod_{k=1}^{N} \frac{\zeta(r_k)\, r_k\, dr_k\, d\varphi_k}{[1-(r_k^2/4a^2)]} \prod_{i<j} \left| \frac{(z_i - z_j)/(2a)}{1-(z_i \bar{z}_j/4a^2)} \right|^2 \right] \tag{4.2}$$

where for $N = 1$ the product $\prod_{i<j}$ must be replaced by 1. We have defined a position-dependent fugacity $\zeta(r) = \zeta[1 - r^2/(4a^2)]^{4\pi n_b a^2 - 1}\, e^C$ which includes the particle-background interaction (2.10) and only one factor $[1 - r^2/(4a^2)]^{-1}$ from the integration measure $dS = [1 - r^2/(4a^2)]^{-2}\, d\mathbf{r}$. This should prove to be convenient later. The e^C factor is

$$e^C = \exp\left[4\pi n_b a^2 \left(\ln \cosh^2 \frac{\tau_0}{2} - \sinh^2 \frac{\tau_0}{2} \ln \tanh^2 \frac{\tau_0}{2} \right) \right] \tag{4.3}$$

which is a constant term coming from the particle-background interaction term (2.10) and

$$\ln C_0 = \frac{(4\pi n_b a^2)^2}{2} \left[\ln \cosh^2 \frac{\tau_0}{2} + \sinh^2 \frac{\tau_0}{2} \left(\sinh^2 \frac{\tau_0}{2} \ln \tanh^2 \frac{\tau_0}{2} - 1 \right) \right] \tag{4.4}$$

which comes from the background-background interaction (2.11). Notice that for large domains, when $\tau_0 \to \infty$, we have

$$e^C \sim \left[\frac{e^{\tau_0 + 1}}{4} \right]^{4\pi n_b a^2} \tag{4.5}$$

and

$$\ln C_0 \sim -\frac{(4\pi n_b a^2)^2\, e^{\tau_0}}{4} \tag{4.6}$$

Let us define a set of reduced complex coordinates $u_i = (z_i/2a)$ inside the Poincaré disk and its corresponding images $u_i^* = (2a/\bar{z}_i)$ outside the disk. By using Cauchy identity

$$\det \left(\frac{1}{u_i - u_j^*} \right)_{(i,j) \in \{1,\ldots,N\}^2} = (-1)^{N(N-1)/2} \frac{\prod_{i<j} (u_i - u_j)(u_i^* - u_j^*)}{\prod_{i,j} (u_i - u_j^*)} \tag{4.7}$$

the particle-particle interaction term together with the $[1-(r_i^2/4a^2)]^{-1}$ other term from the integration measure can be cast into the form

$$\prod_{i<j} \left| \frac{(z_i - z_j)/(2a)}{1-(z_i \bar{z}_j/4a^2)} \right|^2 \prod_{i=1}^{N} [1-(r_i^2/4a^2)]^{-1} = \det \left(\frac{1}{1 - u_i \bar{u}_j} \right)_{(i,j) \in \{1,\ldots,N\}^2} \tag{4.8}$$

The grand canonical partition function then is

$$\Xi = \left[1 + \sum_{N=1}^{\infty} \frac{1}{N!} \int \prod_{k=1}^{N} d^2\mathbf{r}_k\, \zeta(r_k)\, \det\left(\frac{1}{1-u_i\bar{u}_j} \right) \right] C_0 \tag{4.9}$$

We shall now show that this expression can be reduced to an infinite continuous determinant, by using a functional integral representation similar to the one which has been developed for the two-component Coulomb gas.[16] Let us consider the Gaussian partition function

$$Z_0 = \int \mathscr{D}\psi\, \mathscr{D}\bar{\psi}\, \exp\left[\int \bar{\psi}(\mathbf{r})\, M^{-1}(z, \bar{z}')\, \psi(\mathbf{r}')\, d^2\mathbf{r}\, d^2\mathbf{r}' \right] \tag{4.10}$$

The fields ψ and $\bar{\psi}$ are anticommuting Grassmann variables. The Gaussian measure in (4.10) is chosen such that its covariance is equal to[8]

$$\langle \bar{\psi}(\mathbf{r}_i)\, \psi(\mathbf{r}_j) \rangle = M(z_i, \bar{z}_j) = \frac{1}{1-u_i\bar{u}_j} \tag{4.11}$$

where $\langle \cdots \rangle$ denotes an average taken with the Gaussian weight of (4.10). By construction we have

$$Z_0 = \det(M^{-1}) \tag{4.12}$$

Let us now consider the following partition function

$$Z = \int \mathscr{D}\psi\, \mathscr{D}\bar{\psi}\, \exp\left[\int \bar{\psi}(\mathbf{r})\, M^{-1}(z, \bar{z}')\, \psi(\mathbf{r}')\, d^2\mathbf{r}\, d^2\mathbf{r}' + \int \zeta(r)\, \bar{\psi}(\mathbf{r})\, \psi(\mathbf{r})\, d^2\mathbf{r} \right] \tag{4.13}$$

which is equal to

$$Z = \det(M^{-1} + \zeta) \tag{4.14}$$

and then

$$\frac{Z}{Z_0} = \det[M(M^{-1} + \zeta)] = \det[1 + K] \tag{4.15}$$

[8] Actually the operator M should be restricted to act only on analytical functions for its inverse M^{-1} to exist.

where

$$K(\mathbf{r},\mathbf{r}') = M(z,\bar{z}')\,\zeta(r') = \frac{\zeta(r')}{1-u\bar{u}'} \tag{4.16}$$

The results which follow can also be obtained by exchanging the order of the factors M and $M^{-1}+\zeta$ in (4.15), i.e., by replacing $\zeta(r')$ by $\zeta(r)$ in (4.16), however using the definition (4.16) of K is more convenient. Expanding the ratio Z/Z_0 in powers of ζ we have

$$\frac{Z}{Z_0} = 1 + \sum_{N=1}^{\infty} \frac{1}{N!} \int \prod_{k=1}^{N} d^2\mathbf{r}_k\, \zeta(r_k) \langle \bar{\psi}(\mathbf{r}_1)\,\psi(\mathbf{r}_1)\cdots\bar{\psi}(\mathbf{r}_N)\,\psi(\mathbf{r}_N)\rangle \tag{4.17}$$

Now, using Wick theorem for anticommuting variables,[16] we find that

$$\langle \bar{\psi}(\mathbf{r}_1)\,\psi(\mathbf{r}_1)\cdots\bar{\psi}(\mathbf{r}_N)\,\psi(\mathbf{r}_N)\rangle = \det M(z_i,\bar{z}_j) = \det\left(\frac{1}{1-u_i\bar{u}_j}\right) \tag{4.18}$$

Comparing Eqs. (4.17) and (4.9) with the help of Eq. (4.18) we conclude that[9]

$$\Xi = C_0\,\frac{Z}{Z_0} = C_0\,\det(1+K) \tag{4.19}$$

The problem of computing the grand canonical partition function has been reduced to finding the eigenvalues of the operator K. The eigenvalue problem for K reads

$$\int \zeta e^{C}\, \frac{\left(1-\frac{r'^2}{4a^2}\right)^{4\pi n_b a^2-1}}{1-\frac{z\bar{z}'}{4a^2}}\, \Phi(\mathbf{r}')\, r'\,dr'\,d\varphi' = \lambda\Phi(\mathbf{r}) \tag{4.20}$$

For $\lambda \neq 0$ we notice from Eq. (4.20) that $\Phi(\mathbf{r}) = \Phi(z)$ is an analytical function of z. Because of the circular symmetry it is natural to try $\Phi(z) = \Phi_\ell(z) = z^\ell = r^\ell e^{i\ell\varphi}$ with ℓ a non-negative integer (the functions z^ℓ form a complete basis for the analytical functions). Expanding

$$\frac{1}{1-\frac{z\bar{z}'}{4a^2}} = \sum_{n=0}^{\infty}\left(\frac{z\bar{z}'}{4a^2}\right)^n \tag{4.21}$$

[9] Actually, the determinants Z_0 and Z are divergent quantities, since the eigenvalues of M (restricted to act on analytical functions) are easily found to be $4\pi a^2/(\ell+1)$, with ℓ any non-negative integer. However, the ratio Z/Z_0 turns out to be finite.

and replacing $\Phi_\ell(z) = z^\ell$ in Eq. (4.20) one can show that Φ_ℓ is actually an eigenfunction of K with eigenvalue

$$\lambda_\ell = 4\pi a^2 \zeta e^C B_{t_0}(\ell+1, 4\pi n_b a^2) \tag{4.22}$$

with $t_0 = r_0^2/(4a^2) = \tanh^2(\tau_0/2)$ and

$$B_{t_0}(\ell+1, 4\pi n_b a^2) = \int_0^{t_0} (1-t)^{4\pi n_b a^2 - 1}\, t^\ell \, dt \tag{4.23}$$

the incomplete beta function. So we finally arrive to the result for the grand potential

$$\beta\Omega = -\ln \Xi = -\ln C_0 - \sum_{\ell=0}^{\infty} \ln(1 + 4\pi a^2 \zeta e^C B_{t_0}(\ell+1, 4\pi n_b a^2)) \tag{4.24}$$

with e^C and $\ln C_0$ given by Eqs. (4.3) and (4.4). This result is valid for any disk domain of radius $a\tau_0$. Later, in Section 4.3, we will derive a more explicit expression of the grand potential for large domains $\tau_0 \to \infty$.

4.2. The Density

As usual one can compute the density by doing a functional derivative of the grand potential with respect to the position-dependent fugacity:

$$n^{(1)}(\mathbf{r}) = \left(1 - \frac{r^2}{4a^2}\right)^2 \zeta(r) \frac{\delta \ln \Xi}{\delta \zeta(r)} \tag{4.25}$$

The factor $[1-(r^2/4a^2)]^2$ is due to the curvature,[4] so that $n^{(1)}(\mathbf{r})\, dS$ is the average number of particles in the surface element $dS = [1-(r^2/4a^2)]^{-2}\, d\mathbf{r}$. Using a Dirac-like notation, one can formally write

$$\ln \Xi = \mathrm{Tr} \ln(1+K) + \ln C_0 = \int \langle \mathbf{r}|\ln(1+\zeta(r)\, M)\,|\mathbf{r}\rangle \, d\mathbf{r} + \ln C_0 \tag{4.26}$$

Then, doing the functional derivative (4.25), one obtains

$$\begin{aligned} n^{(1)}(\mathbf{r}) &= \left(1 - \frac{r^2}{4a^2}\right)^2 \zeta(r) \,\langle \mathbf{r}|\,(1+K)^{-1}\, M\,|\mathbf{r}\rangle \\ &= 4\pi a \left(1 - \frac{r^2}{4a^2}\right)^2 \zeta(r)\, \tilde{G}(\mathbf{r}, \mathbf{r}) \end{aligned} \tag{4.27}$$

where we have defined $\tilde{G}(\mathbf{r},\mathbf{r}')$ by[10] $\tilde{G}=(1+K)^{-1}M/(4\pi a)$. More explicitly, $\tilde{G}$ is the solution of $(1+K)\,\tilde{G}=M/(4\pi a)$, that is

$$\tilde{G}(\mathbf{r},\mathbf{r}')+\zeta e^{C}\int \tilde{G}(\mathbf{r}'',\mathbf{r}')\,\frac{\left(1-\frac{r''^2}{4a^2}\right)^{4\pi n_b a^2-1}}{1-\frac{z\bar{z}''}{4a^2}}\,d\mathbf{r}''=\frac{1}{4\pi a\left[1-\frac{z\bar{z}'}{4a^2}\right]} \tag{4.28}$$

and the density is given by

$$n^{(1)}(\mathbf{r})=4\pi a\zeta e^{C}\left(1-\frac{r^2}{4a^2}\right)^{4\pi n_b a^2+1}\tilde{G}(\mathbf{r},\mathbf{r}) \tag{4.29}$$

From the integral Eq. (4.28) one can see that $\tilde{G}(\mathbf{r},\mathbf{r}')$ is an analytical function of z. Thus the solution is of the form

$$\tilde{G}(\mathbf{r},\mathbf{r}')=\sum_{\ell=0}^{\infty} a_\ell(\mathbf{r}')\,z^\ell \tag{4.30}$$

and Eq. (4.28) yields

$$\tilde{G}(\mathbf{r},\mathbf{r}')=\frac{1}{4\pi a}\sum_{\ell=0}^{\infty}\left(\frac{z\bar{z}'}{4a^2}\right)^{\ell}\frac{1}{1+4\pi a^2\zeta e^{C}B_{t_0}(\ell+1,4\pi n_b a^2)} \tag{4.31}$$

Then the density is given by

$$n^{(1)}(r)=\zeta e^{C}\left(1-\frac{r^2}{4a^2}\right)^{4\pi n_b a^2+1}\sum_{\ell=0}^{\infty}\left(\frac{r^2}{4a^2}\right)^{\ell}\frac{1}{1+4\pi a^2\zeta e^{C}B_{t_0}(\ell+1,4\pi n_b a^2)} \tag{4.32}$$

After some calculation (see Appendix A), it can be shown that, in the limit $a\to\infty$, the result for the flat disk in the canonical ensemble[17]

$$\frac{n^{(1)}(r)}{n_b}=\exp(-\pi n_b r^2)\sum_{\ell=0}^{N_b-1}\frac{(\pi n_b r^2)^\ell}{\gamma(\ell+1,N_b)} \tag{4.33}$$

is recovered, up to a correction due to the non-equivalence of ensembles in finite systems. In (4.33), γ is the incomplete gamma function

$$\gamma(\ell+1,x)=\int_0^x t^\ell e^{-t}\,dt \tag{4.34}$$

[10] The factor $4\pi a$ is there just to keep the same notations as in ref. 4.

In that flat-disk case, in the thermodynamic limit (half-space), $n^{(1)}(r_0) = n_{\text{contact}} \to n_b \ln 2$.

4.3. Large Domains

We are now interested in large domains $\tau_0 \to \infty$. In this thermodynamic limit we will show that the sums in Eqs. (4.24) and (4.32) can be replaced by integrals. For pedagogical reasons we will first consider the case $4\pi n_b a^2 = 1$ in which the calculations are simpler, and afterwards deal with the general case.

4.3.1. The Case $4\pi n_b a^2=1$

In this case the incomplete beta function that appears in Eqs. (4.24) and (4.32) simply is

$$B_{t_0}(\ell+1, 1) = \frac{t_0^{\ell+1}}{\ell+1} = \frac{[\tanh^2(\tau_0/2)]^{\ell+1}}{\ell+1} \tag{4.35}$$

When $\tau_0 \to \infty$ we have

$$B_{t_0}(\ell+1, 1) \sim \frac{\exp(-4(\ell+1)\, e^{-\tau_0})}{\ell+1} \tag{4.36}$$

Then the sum appearing in the grand potential (4.24) takes the form

$$\sum_{\ell=0}^{\infty} \ln\left(1+\frac{\zeta e}{n_b} \frac{\exp(-4(\ell+1)\, e^{-\tau_0})}{4(\ell+1)\, e^{-\tau_0}}\right) \tag{4.37}$$

where we have used the asymptotic expression (4.5) for e^C. This sum can be seen as a Riemann sum for the variable $x = 4(\ell+1)\, e^{-\tau_0}$. Indeed, for large values of τ_0, the variable x varies in small steps $dx = 4e^{-\tau_0}$. The sum (4.37) then converges, when $\tau_0 \to \infty$, to the integral

$$\int_0^{\infty} \ln\left(1+\frac{\zeta e}{n_b} \frac{e^{-x}}{x}\right) \frac{dx}{4e^{-\tau_0}} \tag{4.38}$$

This expression together with Eq. (4.6) for the constant $\ln C_0$ gives the grand potential in the thermodynamic limit $\tau_0 \to \infty$

$$\beta\Omega \sim -\frac{e^{\tau_0}}{4}\left[\int_0^{\infty} \ln\left(1+\frac{\zeta e}{n_b} \frac{e^{-x}}{x}\right) dx - 1\right] \tag{4.39}$$

We notice that the grand potential is extensive as expected. The area of the system being $\mathscr{A} = 4\pi a^2 \sinh^2(\tau_0/2) \simeq \pi a^2 e^{\tau_0}$, we find that the grand potential per unit area $\omega = \Omega/\mathscr{A}$ is given by

$$\beta\omega = -n_b \left[\int_0^\infty \ln\left(1 + \frac{\zeta e}{n_b}\frac{e^{-x}}{x}\right) dx - 1 \right] \tag{4.40}$$

Similar calculations lead from Eq. (4.32) to the density $n^{(1)}(\sigma)$ near the boundary as a function of the distance from that boundary $a\sigma = a(\tau_0 - \tau)$,

$$n^{(1)}(\sigma) = \zeta e\, e^{2\sigma} \int_0^\infty \frac{e^{-xe^\sigma}}{1 + \frac{\zeta e}{n_b}\frac{e^{-x}}{x}}\, dx \tag{4.41}$$

After the change of variable $xe^\sigma \to x$, this can be written as

$$\frac{n^{(1)}(\sigma)}{n_b} = \int_0^\infty \frac{xe^{-x}\, dx}{\frac{xe^{-\sigma}}{(\zeta e/n_b)} + e^{-xe^{-\sigma}}} \tag{4.42}$$

The average density $n = N/\mathscr{A}$ can be obtained integrating the density profile (4.42) or by using the thermodynamic relation $N = -\beta\zeta(\partial\Omega/\partial\zeta)$. We find

$$\frac{n}{n_b} = \int_0^\infty \frac{e^{-x}\, dx}{\frac{x}{(\zeta e/n_b)} + e^{-x}} \tag{4.43}$$

4.3.2. The General Case

With the case $4\pi n_b a^2 = 1$ we have illustrated the general procedure for computing the thermodynamic limit. Now we proceed to compute it in the more general case where $4\pi n_b a^2$ has any positive value. To simplify the notations let us define $\alpha = 4\pi n_b a^2$. The main difficulty is to find a suitable asymptotic expression of the incomplete beta function

$$B_{t_0}(\ell+1, \alpha) = \int_0^{t_0} (1-t)^{\alpha-1}\, t^\ell\, dt \tag{4.44}$$

when $t_0 \to 1$ which is valid for large ℓ. As we have noticed in the previous section the main contribution from the sum in ℓ that appears in the grand potential comes from large values of ℓ which are of order e^{τ_0}. For these values of ℓ the integrand in the definition of the beta function $(1-t)^{\alpha-1} t^{\ell}$ is very peaked around $t=t_0$ and decays very fast when $t \to 0$. So the main contribution to the incomplete beta function comes from values of t near t_0. It is then natural to do the change of variable in the integral $t=t_0-v$ where with the new variable v the integral is mainly dominated by small values of v. Then we have

$$B_{t_0}(\ell+1, \alpha) = \int_0^{t_0} (1-t_0+v)^{\alpha-1} e^{\ell \ln(t_0-v)} \, dv \tag{4.45}$$

Replacing t_0 by its asymptotic value $t_0 \sim 1-4e^{-\tau_0}$ and taking into account that v is small (of order $e^{-\tau_0}$), we find, at first order in $e^{-\tau_0}$,

$$B_{t_0}(\ell+1, \alpha) \sim \frac{1}{\ell^{\alpha}} \Gamma(\alpha, x) \tag{4.46}$$

where we have introduced once more the variable $x=4\ell e^{-\tau_0}$ (at first order in $e^{-\tau_0}$ it is the same variable $x=4(\ell+1)\, e^{-\tau_0}$ introduced in the case $\alpha=1$) and

$$\Gamma(\alpha, x) = \int_x^{\infty} y^{\alpha-1} e^{-y} \, dy \tag{4.47}$$

is an incomplete gamma function. With this result and Eq. (4.5) the term $e^C B_{t_0}(\ell+1, \alpha)$ in the expressions (4.24) and (4.32) appears as a function of the continuous variable $x=4\ell e^{-\tau_0}$

$$e^C B_{t_0}(\ell+1, \alpha) \sim e^{\alpha} \frac{\Gamma(\alpha, x)}{x^{\alpha}} \tag{4.48}$$

With this result we can replace the sums for ℓ in Eqs. (4.24) and (4.32) by integrals over the variable x and we find the following expressions for the grand potential per unit area

$$\beta\omega = \frac{1}{4\pi a^2} \left\{ (4\pi n_b a^2)^2 - \int_0^{\infty} \ln \left[1 + 4\pi a^2 \zeta e^{4\pi n_b a^2} \frac{\Gamma(4\pi n_b a^2, x)}{x^{4\pi n_b a^2}} \right] dx \right\} \tag{4.49}$$

and the density

$$n^{(1)}(\sigma) = \zeta e^{4\pi n_b a^2} e^{(4\pi n_b a^2+1)\sigma} \int_0^{\infty} \frac{e^{-x e^{\sigma}} \, dx}{1 + 4\pi a^2 \zeta e^{4\pi n_b a^2} \dfrac{\Gamma(4\pi n_b a^2, x)}{x^{4\pi n_b a^2}}} \tag{4.50}$$

In particular the contact value of the density, that is when $\sigma = 0$, is

$$n_{\text{contact}} = n^{(1)}(0) = \zeta e^{4\pi n_b a^2} \int_0^\infty \frac{e^{-x}\,dx}{1 + 4\pi a^2 \zeta e^{4\pi n_b a^2} \dfrac{\Gamma(4\pi n_b a^2, x)}{x^{4\pi n_b a^2}}} \tag{4.51}$$

After some calculation (see Appendix A), it can be shown that, in the limit $a \to \infty$, the result for the flat disk in the thermodynamic limit $n_{\text{contact}} = n_b \ln 2$ is again recovered.

An alternative expression for the density which we will also use is obtained by doing the change of variable $xe^{\sigma} \to x$ and introducing again $\alpha = 4\pi n_b a^2$

$$\frac{n^{(1)}(\sigma)}{n_b} = \int_0^\infty \frac{x^{\alpha} e^{-x}\,dx}{\dfrac{x^{\alpha} e^{-\alpha\sigma}}{(\zeta e^{\alpha}/n_b)} + \alpha \Gamma(\alpha, xe^{-\sigma})} \tag{4.52}$$

From this expression it can be seen that in the bulk, when $\sigma \to \infty$ and $e^{-\sigma} \to 0$, the density is equal to the background density, $n^{(1)}(\sigma) \to n_b$. The system is neutral in the bulk. The excess charge, which is controlled by the fugacity ζ, concentrates as usual on the boundary.

The average total number of particles N and the average density $n = N/\mathscr{A}$ can be computed either by using the thermodynamic relation

$$N = -\beta\zeta \frac{\partial \Omega}{\partial \zeta} \tag{4.53}$$

or by integrating the density profile (4.50)

$$N = \int_{\tau < \tau_0} n^{(1)}(\sigma)\, dS = \pi a^2 e^{\tau_0} \int_0^\infty n^{(1)}(\sigma)\, e^{-\sigma}\, d\sigma \tag{4.54}$$

The two methods yield the same result, as expected,

$$n = \frac{N}{\mathscr{A}} = \zeta e^{4\pi n_b a^2} \int_0^\infty \frac{\Gamma(4\pi n_b a^2, x)\, dx}{x^{4\pi n_b a^2} + 4\pi a^2 \zeta e^{4\pi n_b a^2} \Gamma(4\pi n_b a^2, x)} \tag{4.55}$$

The ratio of the average density and the background density can be put in the form

$$\frac{n}{n_b} = \int_0^\infty \frac{\Gamma(\alpha, x)\, dx}{\dfrac{x^{\alpha}}{(\zeta e^{\alpha}/n_b)} + \alpha \Gamma(\alpha, x)} \tag{4.56}$$

As seen on Eqs. (4.52) and (4.56) the density profile $n^{(1)}(\sigma)$ and the average density n are functions of the parameter $g = \zeta e^{4\pi n_b a^2}/n_b$. Different values of this parameter g give different density profiles and mean densities. Depending on the value of g the system can be globally positive, negative or neutral. From Eq. (4.56) it can be seen that the average density is a monotonous increasing function of the fugacity, as it should be. Therefore there is one unique value of the fugacity for which the system is globally neutral. For the case $4\pi n_b a^2 = 1$, we have determined numerically the value of g needed for the system to be neutral, $n = n_b$. This value is $g = \zeta e/n_b = 1.80237$.

It may be noted that, in the case of a flat disk in the grand canonical ensemble, the 2D OCP remains essentially neutral (the modulus of its total charge cannot exceed one elementary charge q), whatever the fugacity ζ might be;[18, 19] this is because the Coulomb interaction $-\ln(r/L)$ becomes infinite at infinity and bringing an excess charge from a reservoir at infinity to the system already carrying a net charge would cost an infinite energy. On the contrary, in the present case of a 2D OCP on a pseudosphere, the Coulomb interaction (2.8) has an exponential decay at large distances, and varying the fugacity does change the total charge of the disk.

Figure 1 shows several plots of the density $n^{(1)}(\sigma)$ as a function of the distance σ from the boundary (in units of a), for different values of g, in the case $\alpha = 4\pi n_b a^2 = 1$. It is interesting to notice that for $g \leqslant 1$ the density

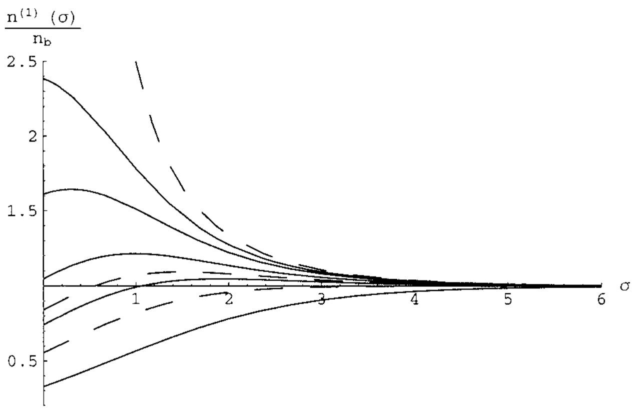

Fig. 1. The density profile $n^{(1)}(\sigma)$ (in units of n_b) as a function of the distance from the boundary σ (in units of a) for different values of the parameter $g = \zeta e/n_b$ in the case $4\pi n_b a^2 = 1$. From bottom to top, in full line $g = 0.5, 1.5, 2.5, 5.0, 10.0$ and in dashed line $g = 1$ (change of behavior between monotonous increasing profile and oscillating profile), $g = 1.80237$ (globally neutral system) and $g \to \infty$.

is always an increasing function of σ. Far away from the boundary, the density approaches the background density n_b from below. On the other hand when $g > 1$, but not too large, the density profile shows an oscillation: $n^{(1)}(\sigma)$ is no longer a monotonous function of σ. Far away from the boundary, $\sigma \to \infty$, the density now approaches the background density from above. Finally, when g is large enough, the density profile is again monotonous, now a decreasing function of σ.

The change of behavior as $\sigma \to \infty$ can actually be shown analytically. Let us define $u = e^{-\sigma}$. From Eq. (4.52) we have

$$\frac{\partial}{\partial u}\left(\frac{n^{(1)}(\sigma)}{n_b}\right) = \int_0^\infty \frac{\alpha x^{2\alpha} u^{\alpha-1} e^{-x-xu}}{\left(\frac{(xu)^\alpha}{g} + \alpha\Gamma(\alpha, xu)\right)^2}\left[1 - \frac{e^{xu}}{g}\right] dx \tag{4.57}$$

The first term in the integral is always positive. The second term, $1-(e^{xu}/g)$, in the limit $\sigma \to \infty$ $(u \to 0)$ is $1-(1/g)$. If $g < 1$ it is negative, then $\partial n^{(1)}/\partial u$ is negative and $n^{(1)}(\sigma)$ is then an increasing function of σ when $\sigma \to \infty$ as it was noticed in the last paragraph.

Also, in this case $\alpha = 1$, when $\zeta \to \infty$ the density profile (4.42) can be computed explicitly

$$\frac{n^{(1)}(\sigma)}{n_b} = \frac{1}{(1-e^{-\sigma})^2} \tag{4.58}$$

It is clearly a monotonous decreasing function of σ.

4.4. Relations Between the Different Pressures

From the explicit expressions (4.49) and (4.52) for the grand potential and the density profile, we can check the relations between the different pressures obtained in Section 3. The mechanical pressure simply is $P^{(m)} = -\omega$ and it is given by Eq. (4.49). This expression can be transformed by doing an integration by parts in the integral giving

$$\beta P^{(m)} = -\frac{1}{4\pi a^2}\left\{\int_0^\infty \frac{4\pi a^2 x\zeta e^{4\pi n_b a^2} \frac{d}{dx}\left[\frac{\Gamma(4\pi n_b a^2, x)}{x^{4\pi n_b a^2}}\right]}{1 + 4\pi a^2 \zeta e^{4\pi n_b a^2} \frac{\Gamma(4\pi n_b a^2, x)}{x^{4\pi n_b a^2}}} dx + (4\pi n_b a^2)^2\right\} \tag{4.59}$$

By the replacement

$$\frac{d}{dx}\left[\frac{\Gamma(4\pi n_b a^2, x)}{x^{4\pi n_b a^2}}\right] = -\frac{e^{-x}}{x} - 4\pi n_b a^2 \frac{\Gamma(4\pi n_b a^2, x)}{x^{4\pi n_b a^2+1}} \tag{4.60}$$

in Eq. (4.59), one recognizes the expressions (4.51) and (4.55) for the contact density and the average density, thus giving

$$\beta P^{(m)} = n^{(1)}(0) + 4\pi n_b a^2 (n - n_b) \tag{4.61}$$

which is precisely, when $\beta q^2 = 2$, the relation (3.25) between the mechanical pressure $P^{(m)}$ and the kinetic pressure $P^{(k)} = (1/\beta)\, n^{(1)}(0)$ obtained in Section 3.

The thermal pressure is

$$P^{(\theta)} = -\omega(\zeta, n_b) + n_b \left(\frac{\partial \omega(\zeta, n_b)}{\partial n_b}\right)_{\zeta} \tag{4.62}$$

The last term in this equation is given by

$$\beta n_b \frac{\partial \omega}{\partial n_b} = \frac{1}{4\pi a^2}\left\{2\alpha^2 - \int_0^\infty \frac{4\pi a^2 \zeta}{1 + \dfrac{4\pi a^2 \zeta e^{\alpha} \Gamma(\alpha, x)}{x^{\alpha}}}\, \alpha \frac{\partial}{\partial \alpha}\left[\frac{e^{\alpha}\Gamma(\alpha, x)}{x^{\alpha}}\right] dx\right\} \tag{4.63}$$

Making the replacement

$$\alpha \frac{\partial}{\partial \alpha}\left[\frac{e^{\alpha}\Gamma(\alpha, x)}{x^{\alpha}}\right] = \alpha e^{\alpha}\left(\frac{\Gamma(\alpha, x)}{x^{\alpha}} + \frac{\partial}{\partial \alpha}\left[\frac{\Gamma(\alpha, x)}{x^{\alpha}}\right]\right) \tag{4.64}$$

in Eq. (4.63), one recognizes in the first term the average density n, thus obtaining

$$\beta n_b \frac{\partial \omega}{\partial n_b} = \alpha(2n_b - n) - \alpha I \tag{4.65}$$

where

$$I = \int_0^\infty \frac{\zeta e^{\alpha}}{1 + \dfrac{4\pi a^2 \zeta e^{\alpha} \Gamma(\alpha, x)}{x^{\alpha}}} \frac{\partial}{\partial \alpha}\left[\frac{\Gamma(\alpha, x)}{x^{\alpha}}\right] dx \tag{4.66}$$

So the thermal pressure is given by

$$\beta P^{(\theta)} = n^{(1)}(0) + \alpha n_b - \alpha I \tag{4.67}$$

On the other hand the integral appearing in the general relation (3.21) between the thermal pressure and the kinetic pressure

$$J = \int_0^\infty (n^{(1)}(\sigma) - n_b)\, e^{-\sigma} \sigma \, d\sigma \tag{4.68}$$

can be split into two parts

$$J = -n_b + I' \tag{4.69}$$

with

$$I' = \int_0^\infty n^{(1)}(\sigma)\, \sigma e^{-\sigma} \, d\sigma \tag{4.70}$$

Using the actual integral representation for the density profile given by Eq. (4.50) yields

$$I' = \int_0^\infty \frac{\zeta e^\alpha}{1 + \dfrac{4\pi a^2 \zeta e^\alpha \Gamma(\alpha, x)}{x^\alpha}} \left\{ \int_0^\infty e^{\alpha\sigma} e^{-xe^\sigma} \sigma \, d\sigma \right\} dx \tag{4.71}$$

The integral over σ can be cast in the form

$$\frac{\partial}{\partial \alpha} \left[\int_0^\infty e^{\alpha\sigma} e^{-xe^\sigma} \, d\sigma \right] \tag{4.72}$$

By doing the change of variable $y = xe^\sigma$ one immediately recognizes the integral representation of the incomplete gamma function. The above expression is then equal to

$$\frac{\partial}{\partial \alpha} \left[\frac{\Gamma(\alpha, x)}{x^\alpha} \right] \tag{4.73}$$

Thus we have proven that $I' = I$ and finally we have the relation

$$\beta(P^{(\theta)} - P^{(k)}) = -4\pi n_b a^2 \int_0^\infty (n^{(1)}(\sigma) - n_b)\, e^{-\sigma} \sigma \, d\sigma \tag{4.74}$$

which is relation (3.21) in the solvable case $\beta q^2 = 2$.

5. CONCLUSION

In a flat space, the neighborhood of the boundary of a large domain has a volume which is a negligible fraction of the whole volume. This is why, for the statistical mechanics of ordinary fluids, usually there is a thermodynamic limit: when the volume becomes infinite, quantities such as the free energy per unit volume or the pressure have a unique limit, independent of the domain shape and of the boundary conditions. However, even in a flat space, the one-component plasma is special. For the OCP, it is possible to define several non-equivalent pressures, some of which, for instance the kinetic pressure, obviously are surface-dependent even in the infinite-system limit.

Even for ordinary fluids, statistical mechanics on a pseudosphere is expected to have special features, which are essentially related to the property that, for a large domain, the area of the neighborhood of the boundary is of the same order of magnitude as the whole area. Although some bulk properties, such as correlation functions far away from the boundary, will exist, extensive quantities such as the free energy or the grand potential are strongly dependent on the boundary neighborhood and surface effects. For instance, in the large-domain limit, no unique limit is expected for the free energy per unit area $F/\mathscr{A}$ or the pressure $-(\partial F/\partial \mathscr{A})_{\beta, N}$.

In the present paper, we have studied the 2D OCP on a pseudosphere, for which surface effects are expected to be important for both reasons: because we are dealing with a one-component plasma and because the space is a pseudosphere. Therefore, although the correlation functions far away from the boundary have unique thermodynamic limits,[4] many other properties are expected to depend on the domain shape and on the boundary conditions. This is why we have considered a special well-defined geometry: the domain is a disk bounded by a plain hard wall, and we have studied the corresponding large-disk limit. Our results have been derived only for that geometry.

We have been especially interested by different pressures which can be defined for this system. It has been shown that the virial pressure $P^{(v)}$ (defined through the virial theorem) and the kinetic pressure $P^{(k)}$ (the force per unit length that the particles alone exert on the wall) are equal to each other. We have also considered the thermal pressure $P^{(\theta)}$, the definition of which includes contributions from the background. It should be noted that this thermal pressure is also dependent on surface effects, since it is defined by (3.14) and (3.15) in terms of the free energy or the grand potential, and the corresponding partition functions include relevant contributions from the surface region. The thermal pressure is not equal to the previous ones. We have also considered the so-called mechanical pressure $P^{(m)}$ which differs from the kinetic one only for charged systems. General relations among these different pressures have been established.

One of the referees of the present paper has asked which one of these different pressures is the "right" one, i.e., which one would be measured by a barometer. The answer, based on the previous paragraph, is that it depends on which kind of "barometer" is used. For instance, the measured pressure would not be the same if the barometer, placed on the wall, measures only the force exerted on it by the particles alone, or if it also feels the force exerted by the background.

When $\beta q^2 = 2$, the model is exactly solvable, in the grand canonical ensemble. Explicit expressions have been obtained for the grand potential, the density profile, and the pressures. The general relations between the different pressures have been checked.

A bulk pressure, independent of the surface effects, can be defined from the Maxwell stress tensor. It is not astonishing that this bulk pressure is different from the previous ones, all of which depend on surface effects.

APPENDIX A. THE FLAT LIMIT

In this Appendix we study the flat limit $a \to \infty$ of the expressions found for the density in Section 4. We shall study the limit $a \to \infty$ for a finite system and then take the thermodynamic limit and compare to the result of taking first the thermodynamic limit and then the flat limit $a \to \infty$. Since for a large system on the pseudosphere boundary effects are of the same order as bulk effects it is not clear a priori whether computing these two limits in different order would give the same results. We shall show that, indeed, the same results are obtained.

For a finite disk of radius $d = a\tau_0$, we have in the flat limit $a \to \infty$, $d \sim r_0$. In Eq. (4.32), in the limit $a \to \infty$, the term e^C given by (4.3) becomes

$$e^C \sim \left(\frac{r_0^2}{4a^2}\right)^{-N_b} e^{N_b} \tag{A.1}$$

where $N_b = \pi n_b r_0^2$ is the number of particles in the background in the flat limit. Since for large a, $t_0 = r_0^2/(4a^2)$ is small, the incomplete beta function in Eq. (4.32) is

$$\begin{aligned} B_{t_0}(\ell+1, \alpha) &= \int_0^{t_0} e^{(\alpha-1)\ln(1-t)}\, t^\ell\, dt \\ &\sim \int_0^{t_0} e^{-(\alpha-1)t}\, t^\ell\, dt \sim \frac{\gamma(\ell+1, N_b)}{\alpha^{\ell+1}} \end{aligned} \tag{A.2}$$

Expanding $(1-(r^2/4a^2))^{4\pi n_b a^2} \sim \exp(-\pi n_b r^2)$ in Eq. (4.32) we finally find the density as a function of the distance r from the center

$$n^{(1)}(r) = n_b e^{-\pi n_b r^2} \sum_{\ell=0}^{\infty} \frac{(\pi n_b r^2)^\ell}{\alpha^{\ell-N_b} N_b^{N_b} e^{-N_b}(n_b/\zeta) + \gamma(\ell+1, N_b)} \tag{A.3}$$

When $\alpha \to \infty$ the terms for $\ell > N_b$ in the sum vanish because $\alpha^{\ell-N_b} \to \infty$. Then

$$n^{(1)}(r) = n_b e^{-\pi n_b r^2} \sum_{\ell=0}^{E(N_b)-1} \frac{(\pi n_b r^2)^\ell}{\gamma(\ell+1, N_b)} + \Delta n^{(1)}(r) \tag{A.4}$$

The first term is the density for a flat OCP in the canonical ensemble with a background with $E(N_b)$ elementary charges ($E(N_b)$ is the integer part of N_b). The second term is a correction due to the inequivalence of the ensembles for finite systems and it depends on whether N_b is an integer or not. If N_b is not an integer

$$\Delta n^{(1)}(r) = n_b \frac{(\pi n_b r^2)^{E(N_b)} e^{-\pi n_b r^2}}{\gamma(E(N_b)+1, N_b)} \tag{A.5}$$

and if N_b is an integer

$$\Delta n^{(1)}(r) = n_b \frac{(\pi n_b r^2)^{N_b} e^{-\pi n_b r^2}}{N_b^{N_b} e^{-N_b}(n_b/\zeta) + \gamma(N_b+1, N_b)} \tag{A.6}$$

In any case in the thermodynamic limit $r_0 \to \infty$, $N_b \to \infty$, this term $\Delta n^{(1)}(r)$ vanishes giving the known results for the OCP in a flat space in the canonical ensemble.[3, 17] Integrating the profile density (A.4) one finds the average number of particles. For a finite system it is interesting to notice that the average total number of particles N is

$$N = E(N_b) + 1 \tag{A.7}$$

for N_b not an integer and

$$N = N_b + \frac{1}{1 + \dfrac{N_b^{N_b} e^{-N_b} n_b}{\zeta \gamma(N_b+1, N_b)}} \tag{A.8}$$

for N_b an integer. In both cases the departure from the neutral case $N = N_b$ is at most of one elementary charge as it was noticed before.[18, 19]

Let us now consider the other order of the limits. We start with the expression (4.51) for the contact density in the thermodynamic limit in the pseudosphere and show that in the limit $a \to \infty$ the value of the contact density reduces to the known expression for a neutral OCP in a flat space at a hard wall.[17] We also show that in that limit the average density is independent of the fugacity and equal to the background density $n = n_b$.

Equation (4.51) can be rewritten as

$$\frac{n_{\text{contact}}}{n_b} = \int_0^\infty \frac{x^\alpha e^{-x}\,dx}{\frac{n_b}{\zeta}\,x^\alpha e^{-\alpha} + \alpha\Gamma(\alpha, x)} \tag{A.9}$$

For large α, the numerator of the integrand in (A.9) has a sharp peak at $x = \alpha$ and can be expanded as

$$x^\alpha e^{-x} \sim e^{\alpha \ln \alpha - \alpha - \left(\frac{x-\alpha}{\sqrt{2\alpha}}\right)^2} \tag{A.10}$$

In the denominator, using the large α expansion of the incomplete gamma function,[20] and neglecting 1 with respect to α, we obtain

$$\alpha\Gamma(\alpha, x) \sim \alpha^\alpha e^{-\alpha} \sqrt{\frac{\pi\alpha}{2}} \left[1 - \operatorname{erf}\left(\frac{x-\alpha+1}{\sqrt{2\alpha}}\right)\right] \tag{A.11}$$

where

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\,du \tag{A.12}$$

is the error function. Using (A.10) and (A.11) in (A.9) gives

$$\frac{n_{\text{contact}}}{n_b} \sim \int_0^\infty \frac{e^{-\left(\frac{x-\alpha}{\sqrt{2\alpha}}\right)^2}\,dx}{\frac{n_b}{\zeta}\left(\frac{x}{\alpha}\right)^\alpha + \sqrt{\frac{\pi\alpha}{2}}\left[1 - \operatorname{erf}\left(\frac{x-\alpha+1}{\sqrt{2\alpha}}\right)\right]} \tag{A.13}$$

For $x > \alpha$, the first term in the denominator goes to infinity for large α and the integrand goes to zero. On the other hand, when $x < \alpha$, this same first term goes to zero, thus, after the change of variable $t = (x - \alpha)/\sqrt{2\alpha}$,

$$\frac{n_{\text{contact}}}{n_b} \sim \frac{2}{\sqrt{\pi}} \int_{-\sqrt{\alpha/2}}^0 \frac{e^{-t^2}\,dt}{1 - \operatorname{erf}\left(t + \frac{1}{\sqrt{2\alpha}}\right)} \tag{A.14}$$

Finally, as $\alpha \to \infty$,

$$\frac{n_{\text{contact}}}{n_b} \to \int_{-\infty}^{0} \frac{\frac{d\,\text{erf}(t)}{dt}}{1-\text{erf}(t)}\,dt = \ln 2 \tag{A.15}$$

This is the known value[17] for the contact density at a hard plain wall for a neutral OCP.

Following the same lines, Eq. (4.56) for the average density becomes in the limit $\alpha \to \infty$

$$\frac{n}{n_b} \sim \sqrt{\frac{2}{\alpha}} \int_{-\sqrt{\alpha/2}}^{0} \frac{[1-\text{erf}(t)]\,dt}{1-\text{erf}(t)} = 1 \tag{A.16}$$

The average density is equal to the background density and it is independent of the fugacity. Whatever value the fugacity has, the system cannot be charged in the flat case in the thermodynamic limit.

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