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Eliminating nonrenormalizability helps prove
scaled affine quantization of φ_4^4 is nontrivial

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Following a modest comparison between canonical and affine quantization, which points to positive features in the affine procedures. We prove through Monte Carlo analysis that the covariant Euclidean scalar field theory, φ_n^r, where r denotes the power of the interaction term and $n = s + 1$ where s is the spatial dimension and 1 adds imaginary time, such that $r = n = 4$ can be acceptably quantized using scaled affine quantization and the resulting theory is nontrivial, unlike what happens using canonical quantization.

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1. Introduction

Covariant Euclidean scalar field quantization, henceforth denoted φ_n^r, where r is the power of the interaction term and $n = s + 1$ where s is the spatial dimension and 1 adds imaginary time, such that $r < 2n/(n - 2)$ can be treated by canonical quantization (CQ), while models such that $r > 2n/(n - 2)$ are trivial.[1–5] Models such as $r = 2n/(n - 2)$, e.g. $r = n = 4$, also are nonrenormalizable using

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canonical quantization.[1] However, there exists a different approach called affine quantization (AQ)[6,7] that promotes a different set of classical variables to become the basic quantum operators and it offers different results, such as models for which $r > 2n/(n-2)$, which has been recently correctly quantized φ_3^{12}.[8] In this work, we show, with the aid of a Monte Carlo (MC) analysis, that one of the special cases where $r = 2n/(n-2)$, specifically the case $r = n = 4$, can be acceptably quantized using affine quantization.[9–11]

This program was already carried on with partial success in Refs. 9 and 10, where, however, a diverging value of the vacuum expectation value of the field was found. We show here that using a simple rescaling of the affine quantized theory allows to solve this shortcoming keeping the field theory nontrivial.

2. A Comparison Between Canonical Quantization and Affine Quantization for Fields

2.1. *Canonical quantization of scalar fields*

Let us begin with the classical Hamiltonian for a single field $\varphi(x)$

$$H(\pi, \varphi) = \int \left\{ \frac{1}{2} \left[\pi(x)^2 + (\nabla\varphi(x))^2 + m^2\varphi(x)^2 \right] + g\varphi(x)^r \right\} d^s x \,, \qquad (1)$$

where $n = s + 1$ is the number of space–time variables, and r is a positive, even, integer. When g is zero, the remaining expression involves a domain in which a full set of variables, i.e. $\pi(x)$ and $\varphi(x)$, leads to a finite Hamiltonian value. If $g = 0 \to g > 0$, there are two possible results. If $r < 2n/(n-2)$, then the domain remains the same. However, if $r \geq 2n/(n-2)$, then there is a new domain that is smaller than the original domain because the interaction term $\int \varphi(x)^r \, d^s x = \infty$ leads to a reduction of certain fields. The fields that cause that divergence are not $\varphi(x) = \infty$, because that would have eliminated the original domain when $g = 0$. The only way for $\int \varphi(x)^r \, d^s x = \infty$ is, for example, given by $\varphi(x) = 1/[(x-c)^2]^k$ where k is small enough so that the gradient term will diverge sooner than the mass term, while $r > 2$ is big enough so that $\int \varphi(x)^r \, d^s x = \infty$. Such behavior leads to immediate results in perturbation infinities in a power series of g, leading to a nonrenormalizable process, for which quantum efforts, using canonical quantization, collapse to "free" results, despite that $g > 0$, as all that is continuously connected to the original free theory where $g = 0$.

This analysis is confirmed with several efforts. As examples, we note that MC and analytical methods have confirmed that the model φ_4^4 leads only to "free" results,[1–4] as well as the model φ_3^{12} also leads to "free" results.[8] Having seen what CQ can show us what it can do, now let us turn to AQ.

2.2. *Affine quantization of scalar fields*

The classical affine variables are $\kappa(x) \equiv \pi(x)\varphi(x)$ and $\varphi(x) \neq 0$. The reason we insist that $\varphi(x) \neq 0$ is because if $\varphi(x) = 0$ then $\kappa(x) = 0$ and $\pi(x)$ cannot help.

We next introduce the same classical Hamiltonian we chose before now expressed in affine variables. This leads us to

$$H'(\kappa, \varphi) = \int \left\{ \frac{1}{2} \left[\kappa(x)^2 \varphi(x)^{-2} + (\nabla\varphi(x))^2 + m^2\varphi(x)^2 \right] + g\varphi(x)^r \right\} d^s x \,, \quad (2)$$

in which $\varphi(x) \neq 0$ is an important fact. With these variables we do not let $\varphi(x) = \infty$ for the reasons made in the CQ story, but now we must forbid $\varphi(x) = 0$ which would admit $\varphi(x)^{-2} = \infty$. The fact that $0 < \varphi(x)^{-2} < \infty$, it follows that, using these variables, $0 < \varphi(x)^r < \infty$, with any $2 < r < \infty$. This essential result leads to the fact that these AQ bounds on $\varphi(x)$ *forbid any nonrenormalizability*, a "disease" which plagues the CQ analysis. With AQ, this new insight implies that every model φ_n^r does not become a "free" result, but leads to an appropriate "nonfree" result. Specifically, this assertion should lead to "nonfree" results for φ_3^{12} and φ_4^4, as MC results, have already shown.[8–11]

What follows in the coming sections is additional MC studies using AQ procedures. As the former story promises, that study will definitely succeed.

3. Lattice Formulation of the Field Theory

We used a lattice formulation of the AQ field theory studied in Eq. (8) of Ref. 9 using the scaling $\varphi \to a^{-s/2}\varphi$, $g \to a^s g$, $\epsilon \to a^{-s}\epsilon$ where ϵ is the regularization parameter. The theory considers a real scalar field φ taking the value $\varphi(x)$ on each site of a periodic, hypercubic, n-dimensional lattice of lattice spacing a, our ultraviolet cutoff and periodicity $L = Na$. The affine action for the field, $S' = \int H' dx_0$ (with $x_0 = ct$ where c is the speed of light constant and t is imaginary time), is then approximated by

$$\frac{S'[\varphi]}{a^{n-s}} \approx \frac{1}{2} \left\{ \sum_{x,\mu} a^{-2} [\varphi(x) - \varphi(x + e_\mu)]^2 + m^2 \sum_x \varphi(x)^2 \right\}$$

$$+ \sum_x g\,\varphi(x)^r + \frac{3}{8} \sum_x \hbar^2 \frac{1}{\varphi(x)^2 + \epsilon} \,, \quad (3)$$

where e_μ is a vector of length a in the $+\mu$ direction.

In this work, we are interested in reaching the continuum limit by taking Na fixed and seeking $N \to \infty$ at fixed volume L^s and absolute temperature $T = 1/k_B L$ with k_B the Boltzmann's constant.

3.1. *MC results*

We repeated the path integral MC[12–15] calculation for the AQ field theory previously done in Ref. 9 for the case $r = n = 4$ using now the scaling $\varphi \to a^{-s/2}\varphi$,
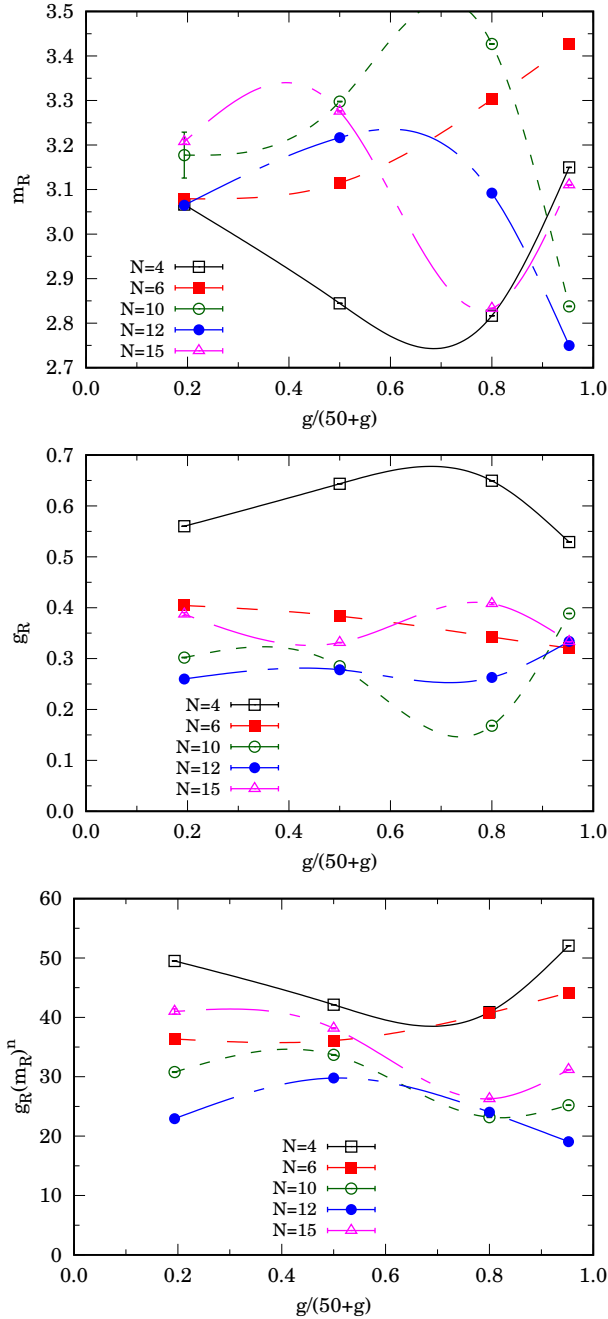
Fig. 1. The renormalized mass $m_R \approx 3$ (top panel), the renormalized coupling constants g_R (central panel), and $g_R m_R^n$ (bottom panel) for various values of the bare coupling constant g at decreasing values of the lattice spacing $a = 1/N$ ($N \to \infty$ idealized limit) for the scaled *affine* φ_4^4 covariant Euclidean scalar field theory described by the action in Eq. (3) for $r = n = 4$. The lines connecting the simulation points are just a guide for the eye.

$g \to a^s g$, $\epsilon \to a^{-s}\epsilon$, which brings to using the lattice formulation for the action of Eq. (3). In particular, we calculated the renormalized coupling constant g_R and mass m_R defined in Eqs. (11) and (13) of Ref. 9, respectively.

Following Freedman *et al.*,[1] for each N and g, we adjusted the bare mass m in such a way to maintain the renormalized mass approximately constant $m_R \approx 3$,[a] to within a few percent (in all cases less than 10%), and we measured the renormalized coupling constant g_R defined in Refs. 8 and 9 for various values of the bare coupling constant g at a given small value of the lattice spacing $a = 1/N$ (this corresponds to choosing an absolute temperature $k_B T = 1$ and a fixed volume $L^3 = 1$). With Na and m_R fixed, as a was made smaller, whatever change we found in $g_R m_R^n$ as a function of g could only be due to the change in a. We generally found that a depression in m_R produced an elevation in the corresponding value of g_R and vice versa. The results are shown in Fig. 1 for the scaled affine action (3) in natural units $c = \hbar = k_B = 1$ and $\epsilon = 10^{-10}$ (the results are independent from the regularization parameter as long as this is chosen sufficiently small), where, following Freedman *et al.*,[1] we decided to compress the range of g for display, by choosing the horizontal axis to be $g/(50 + g)$. The constraint $m_R \approx 3$ was not easy to implement since for each N and g we had to run the simulation several times with different values of the bare mass m in order to determine the value which would satisfy the constraint $m_R \approx 3$.

These results should be compared with the results of Fig. 1 of Freedman *et al.*[1] where the same calculation was done for the canonical version of the field theory. As we can see from our figure, contrary to the figure of Freedman, the renormalized coupling constant of the affine version remains far from zero in the continuum limit when the ultraviolet cutoff is removed ($Na = 1$ and $N \to \infty$) for all values of the bare coupling constant. Here, unlike in the canonical version used by Freedman, the diminishing space between higher N curves is a pointer toward a nonfree ultimate behavior as $N \to \infty$ at fixed volume. Moreover as one can see the $N = 15$ results for the renormalized coupling fall above the ones for $N = 12$.

During our simulations we kept under control also the vacuum expectation value of the field which in all cases was found to vanish in agreement with the fact that the symmetry $\varphi \to -\varphi$ is preserved.

4. Conclusions

In conclusion, we performed a path integral Monte Carlo study of the properties (mass and coupling constant) of the renormalized covariant Euclidean scalar field theory φ_4^4 quantized through scaled affine quantization. As shown in Ref. 11 the vacuum expectation values for the field and the two-point function are well defined. We show here that, unlike what happens for the theory quantized through canonical

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[a]Differently from our previous study[9] with the unscaled version of the affine field theory we did not need to choose complex m in order to fulfill this constraint. Moreover, the needed m was only very slightly depending on g.

quantization, the renormalized coupling constant g_R does not tend to vanish in the continuum limit, where we remove the ultraviolet cutoff at fixed volume. This success of affine quantization to produce a well defined, renormalizable, nontrivial, "nonfree" quantum field theory is one of its merits and benefits.

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