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Statistical Gravity, ADM Splitting, and Affine Quantization

Riccardo Fantoni*

Università di Trieste, Dipartimento di Fisica, strada Costiera 11, Grignano (Trieste), 34151 Italy

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Abstract—In a recent publications I proposed a new statistical theory of gravity [Riccardo Fantoni, Quantum Reports **6**, 706 (2024)], which describes fluctuations of the space-time metric through a *virial temperature*. In a succeeding publication I discussed the foundations [Riccardo Fantoni, Stats **8**, 23 (2025)] of such a theory. Here, I propose a possible way to render numerically accessible the path integral Monte Carlo computations required in the Statistical Gravity theory. This requires the use of the Arnowitt–Deser–Misner (ADM) splitting and of the Affine Quantization (AQ) method.

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1. INTRODUCTION

The idea to realize a quantum theory of gravity has a long history [1, 2]. Recently we proposed a theory for *Statistical Gravity* [3], the FEBB. Leaving aside the feasible experimental confirmations for it, it is yet important to prove that it gives rise to quantities (observables thermal averages) that are mathematically well defined and can therefore be computed (at least numerically). We are thinking, for example, on the problems that one may encounter in computing a *constrained* quantum field theory [4–17], even such a simplest one as the scalar (relativistic Euclidean) theory. In these cases we could experience how important it was to use the method of *Affine Quantization* (AQ) (as opposed to the canonical quantization) in order to render a particular theory *nontrivial*. But even before worrying about renormalizability of a particular quantum field theory, it makes sense to worry about the soundness of the place it occupies in the underlying Hilbert space.

With this in mind, in this short paper, following the idea already put forward in [12] for a construction of a well-defined Quantum Gravity, we propose to use the method of AQ also to construct a well-defined Statistical Gravity.

In these complex tensorial quantum field theories, even the determination of the relevant *semiclassical action* can become a formidable task due to the intertwining of the tensorial calculus and the commutation calculus. Here we will not carry out any such necessary complex calculus explicitly, but we will just lay down the problem showing that it is a well-defined one.

This work is the last of a trilogy on our novel statistical theory of gravity [3, 18].

*E-mail: riccardo.fantoni@scuola.istruzione.it

2. EINSTEIN'S FIELD EQUATIONS FROM A VARIATIONAL PRINCIPLE

Sempre caro mi fu quest'ermo colle,
e questa siepe, che da tanta parte
dell'ultimo orizzonte il guardo esclude.

Giacomo Leopardi
L' Infinito

The Einstein–Hilbert action in general relativity is the action that yields the Einstein field equations through the stationary action principle. With the $(-+++)$ metric signature, the action is given as [19, 20]

$$S = \int \left(\frac{1}{2\kappa}R + \mathcal{L}_{\rm F}\right)\sqrt{-g}\,{\rm d}^4x, \qquad (1)$$

where $g \equiv \det(g_{\mu\nu})$ is the determinant of the metric tensor matrix, $\sqrt{-g}$ is a scalar density, $x \equiv (ct, \mathbf{x})$ is an event with $t \equiv x^0/c$ time and $\mathbf{x} \equiv (x^1, x^2, x^3)$ a point in space, $\sqrt{-g}\,{\rm d}^4x$ is the invariant "volume" element, R is the Ricci scalar, $\kappa = 8\pi G c^{-4}$ is the Einstein gravitational constant (G is the gravitational constant and c is the speed of light in vacuum), and $\sqrt{-g}\,\mathcal{L}_F$ is a Lagrangian density of "interaction" containing the contribution from matter, electromagnetic, or other gauge boson fields to the action. If it converges, the integral is taken over the whole space-time. If it does not converge, S is no longer well-defined, but a modified definition where one integrates over arbitrarily large, relatively compact domains, still yields the Einstein equation as the Euler–Lagrange equation of the Einstein–Hilbert action.

The action was proposed [19] by David Hilbert in 1915 as part of his application of the variational principle $\delta S = 0$ to a combination of gravity and matter, electromagnetism, or other gauge boson fields. Note that in the variation of the Ricci scalar one needs to assume that the Gibbons–Hawking–York boundary term [21–23] gives no contribution to the variation of the action, which is justified at events not in the closure of the boundary, when variation of the metric vanishes in a neighborhood of the boundary or when there is no boundary.

The equations of motion coming from the stationary action principle then read (see the second section of [3])

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{2}$$

which are the Einstein field equations, where

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{F}})}{\delta g^{\mu\nu}} = -2\frac{\delta \mathcal{L}_{\mathrm{F}}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\mathrm{F}} \tag{3}$$

is the stress-energy tensor, and $\kappa = 8\pi G/c^4$ has been chosen such that the nonrelativistic limit yields the usual form of Newton's gravity law.

3. ADM 3 + 1 FOLIATION OF SPACE-TIME

> Ma sedendo e mirando, interminati
> spazi di là da quella, e sovrumani
> silenzi, e profondissima quiete
> io nel pensier mi fingo, ove per poco
> il cor non si spaura.
>
> *Giacomo Leopardi*
> *L' Infinito*

Arnowitt, Deser, and Misner (ADM) proposed in 1962 the following 3 + 1 foliation of space-time [24]:

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{4}$$

where now Latin indexes run over the three spatial components 1, 2, 3. They called N the *lapse* and N_i the *shift*. To split the time component from the 3 spatial components, they chose the following:

$$||g_{\mu\nu}|| = \begin{pmatrix} -(N^2 - N^i N_i) & N_i \\ N_i & g_{ij} \end{pmatrix}, \tag{5}$$

$$||g^{\mu\nu}|| = \begin{pmatrix} -1/N^2 & N^i/N^2 \\ N^i/N^2 & g^{ij} - N^i N^j/N^2 \end{pmatrix}, \tag{6}$$

which are inverse by sight. Note also that $\sqrt{-{}^4g} = N\sqrt{{}^3g}$, where ${}^3g = \det\{g_{ij}\}$, ${}^4g = \det\{g_{\mu\nu}\}$, and we indicate with a presuperscript 4 the full four-dimensional tensor and with a presuperscript 3 the spatial 3×3 tensor, when it is strictly necessary to avoid confusion. Therefore, we will raise (or lower) Greek indices with the full metric tensor $g^{\mu\nu}$ and Latin indices with the spatial metric tensor g^{ij}, which also satisfies $g_{ik}g^{kj} = \delta_i^j$.

ADM showed that if one chooses as the generalized coordinate g_{ij} and as the conjugated momentum

$$\pi^{ij} \equiv \sqrt{-{}^4g}(\Gamma_p{}^0{}_q - g_{pq}\Gamma_r{}^0{}_s g^{rs}) g^{ip} g^{jq}, \tag{7}$$

then the space-time metric Lagrangian is

$$\begin{aligned} \mathcal{L} \equiv \sqrt{-{}^4g}\,{}^4R = -g_{ij}\pi^{ij}{}_{,0} - NR^0 - N_i R^i \\ - 2\left(\pi^{ij}N_j - \frac{1}{2}\pi N^i + N^{|j}\sqrt{{}^3g}\right)_{,i}, \end{aligned} \tag{8}$$

where we denote with a bar (|) a spatial covariant derivative, and

$$R^0 \equiv -\sqrt{{}^3g}\left[{}^3R + \frac{1}{{}^3g}\left(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}\right)\right], \tag{9}$$

$$R^i \equiv -2\pi^{ij}{}_{|j}, \tag{10}$$

$$\pi \equiv \pi_i^i. \tag{11}$$

Equation (9) is the Hamiltonian constraint, whereas Eq. (10) is the momentum constraint. In fact, since the last term in Eq. (8) only contributes a "surface" term to the metric action $S \propto \int \mathcal{L}\, d^4x$, if space-time extends to infinity, it can be taken as giving a negligible contribution.

Taking variations with respect to the lapse and shift provides the constraint equations $R^0 = 0$ and $R^i = 0$, and then the lapse and shift themselves can be freely specified, reflecting the fact that coordinates systems can be freely specified in both space and time.

Since g_{ij} is a strictly positive-definite tensor, in our recent paper [12] we proposed to use affine variables in place of the canonical variables g_{ij} and π^{ij} in order to cure such an *unholonomous* constraint. We then introduce a "dilation" conjugate variable $\pi_j^i = g_{kj}\pi^{ik}$. This classical *momentric* (a name that is a combination of momentum and metric and was invented by John Klauder) tensor and the spatial metric tensor become the new basic canonical affine variables. By doing so and recalling that $g^{ij}{}_{|k} = 0$, we reach the following classical Lagrangian:

$$\mathcal{L} = -g_{ij}\pi^{ij}{}_{,0} - NR^0 - N_i R^i, \tag{12}$$

$$R^i = -2g^{ik}\pi_k{}^j{}_{|j}, \tag{13}$$

$$R^0 = \frac{1}{\sqrt{{}^3g}}\left[\pi_j^i\pi_i^j - \frac{1}{2}\pi^2\right] - \sqrt{{}^3g}\,{}^3R, \tag{14}$$

where we dropped the gradient term in the Lagrangian since it gives no contribution to the classical action,

$$S = \int_0 \int_\Omega \{-g_{ij}\pi^{ij}{}_{,0} - NR^0 - N_i R^i\} \times d(ct)\, d^3\mathbf{x}, \tag{15}$$

where Ω is the region of space and time that starts from the beginning at $t = 0$.

In Affine Quantization (AQ) we promote the two canonical affine variables g_{ij} and π^i_j to operators $\hat{g}_{ij}$ and $\hat{\pi}^i_j$ and write the corresponding affine semiclassical (including just terms up to order $\hbar$ in the $\hbar \to 0$ limit) Lagrangian $\mathcal{L}'$ using the commutation relations between the spatial metric operator and the momentric operator (given, for example, in [12] and derived again in the Appendix).

4. PATH INTEGRAL FORMULATION OF STATISTICAL GRAVITY

> E come il vento
> odo stormir tra queste piante, io quello
> infinito silenzio a questa voce
> vo comparando: e mi sovvien l'eterno,
> e le morte stagioni, e la presente
> e viva, e il suon di lei.
>
> *Giacomo Leopardi*
> *L' Infinito*

Then the action for Einstein's theory of general relativity is one for a particular field theory where the field is the metric tensor $g_{\mu\nu}(x)$, a symmetric tensor with 10 independent components, each of which is a smooth function of 4 variables. We will indicate all these components with the notation $\{g\}(x)$. We will also work in Euclidean time $x^0 \equiv ct \to ict$, so that the metric signature becomes $(+ + ++)$. This amounts to a Wick rotation, which brings from quantum to statistical theory.

The thermal average of an observable $\mathcal{O}[\{g\}(x)]$ will then be given by the following expression [3]:

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{O}[\{g\}(x)] \exp(-\upsilon S')\, \mathcal{D}^{10}\{g\}(x)}{\int \exp(-\upsilon S)\, \mathcal{D}^{10}\{g\}(x)}, \tag{16}$$

so that $\langle 1 \rangle = 1$. Here S' is the affine action

$$S' = \int_0^\beta \int_\Omega \left\{ \frac{1}{2\kappa}\mathcal{L}' + \mathcal{L}2_F N \sqrt{{}^3g} \right\} d(ct)\, d^3\mathbf{x}, \tag{17}$$

where $1/\upsilon$ is a positive constant of dimension of energy times length, $ct \in [0, \beta[$, where $\beta = 1/\tilde{k}_B\tilde{T}$, $\tilde{k}_B$ is the Boltzmann constant of dimensions of one divided by length and by Kelvin degree, and $\tilde{T}$ an *effective* temperature in Kelvin degree (which can be made a field [3], $\tilde{T}(\mathbf{x})$).

Since the thermal average involves taking a trace, we must have $g_{\mu\nu}(ct + \beta, \mathbf{x}) = g_{\mu\nu}(ct, \mathbf{x})$. We will also require periodic spatial boundary conditions on the finite volume $\Omega \subset \mathbb{R}^3$, which is the closest thing to a formal thermodynamic limit. As usual, we will use

$$\mathcal{D}^{10}\{g\}(x) \equiv \prod_x d^{10}\{g\}(x),$$

and the functional integrals will be calculated on a lattice using the path integral Monte Carlo (PIMC) method [25]. Moreover, we will choose

$$d^{10}\{g\}(x) \equiv \prod_{\mu \le \nu} dg^{\mu\nu}(x),$$

where the 10-dimensional space of the 10 independent components of the symmetric metric tensor is assumed to be flat.

The determination of $\mathcal{L}'$ looks like a formidable task that needs to take care of the commutation relations among the spatial metric and the momentric operators, but it seems to be necessary to overcome the numerical singularities that may arise from the *geometrical unholonomous constraint* of having a strictly positive-definite spatial metric. Here we are thinking of a possible loss of ergodicity in the PIMC as its paths wander through and explore the accessible region delimited by sharp constraints which can be variously intricate. We see AQ as a way to smooth out the geometric constraints so as to recover ergodicity and be able to sample the whole relevant region efficiently.

5. CONCLUSIONS

> Così tra questa
> immensità s'annega il pensier mio:
> e il naufragar m'è dolce in questo mare.
>
> *Giacomo Leopardi*
> *L' Infinito*

In this short paper we present a plausible representation (realization) of the FEBB defined in [3]. This requires the use of ADM $3 + 1$ splitting and the AQ procedure. We just lay down the representation but without finding its explicit form which would require rather a formidable calculus, where one needs to deal with commutation relations among tensorial objects. We believe that a Monte Carlo algorithm may lose ergodicity in the presence of sharp constraints, which AQ can otherwise smooth out.

Alternative ways to reach a statistical theory of gravity have been proposed in the past [26–29], but, for example, the use of the Ashtekar canonical variables [26] seems unable to deal with zeroes and infinities as our affine quantization does [30].

Appendix

COMMUTATORS BETWEEN THE SPATIAL METRIC AND THE MOMENTRIC

We start from the Poisson brackets (at fixed time) between the two canonical variables g_{ij} and π^{ij}:

$$\{g_{ij}(x), g_{kl}(x')\} = 0, \tag{A.1}$$

$$\{g_{ij}(x), \pi^{kl}(x')\} = \frac{\delta g_{ij}(x)}{\delta g_{mn}(x'')}\frac{\delta \pi^{kl}(x')}{\delta \pi^{mn}(x'')}$$
$$= \frac{1}{2}\delta^3(x-x'')\delta^3(x'-x'')\delta^k_m\delta^l_n[\delta^m_i\delta^n_j + \delta^m_j\delta^n_i]$$
$$= \frac{1}{2}\delta^3(\mathbf{x}-\mathbf{x}')[\delta^k_i\delta^l_j + \delta^l_i\delta^k_j], \tag{A.2}$$

$$\{\pi^{ij}(x), \pi^{kl}(x')\} = 0, \tag{A.3}$$

where in the second equation we used the symmetry of the metric tensor to write $g_{ij} = [g_{ij} + g_{ji}]/2$, and δ^3 is the three-dimensional Dirac delta function.

We then find the Poisson brackets between the two canonical affine variables g_{ij} and $\pi^j_i = g_{ik}\pi^{kj}$:

$$\{g_{ij}(x), \pi^l_k(x')\} = \{g_{ij}(x), g_{kn}(x')\pi^{nl}(x')\}$$
$$= g_{kn}(x')\{g_{ij}(x), \pi^{nl}(x')\}$$
$$= \frac{1}{2}\delta^3(\mathbf{x}-\mathbf{x}')[\delta^l_j g_{ki}(x) + \delta^l_i g_{kj}(x)], \tag{A.4}$$

$$\{\pi^j_i(x), \pi^l_k(x')\} = \{g_{in}(x)\pi^{nj}(x), g_{km}(x')\pi^{ml}(x')\}$$
$$= g_{km}\pi^{nj}\{g_{in}(x), \pi^{ml}(x')\}$$
$$- g_{in}\pi^{ml}\{g_{km}(x'), \pi^{nj}(x)\}$$
$$= \frac{1}{2}\delta^3(\mathbf{x}-\mathbf{x}')[\delta^l_i\pi^j_k(x) - \delta^j_k\pi^l_i(x)]. \tag{A.5}$$

And in the end we pass to operator commutators, promoted from the Poisson brackets,

$$\{\ldots,\ldots\} \to [\ldots,\ldots]/(i\hbar).$$

After being smeared with suitable test functions, the result is that both the metric and the momentric tensors can be made self-adjoint operators (for example, choosing for the momentric $(\hat{g}_{ik}\hat{\pi}^{jk} + \hat{\pi}^{jk}\hat{g}_{ik})/2$), and the metric operators will satisfy the necessary positivity requirements.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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