Berry’s Geometric Phase

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Quantal phase factors accompanying adiabatic changes

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(Received 13 June 1983)

- Very simple concept, nonetheless missed by the founding fathers of QM in the 1920s and 1930s
- Nowadays in any modern elementary QM textbook
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SUPPLEMENT I

Adiabatic Change and Geometrical Phase

When the author died in 1982, this book was left in manuscript form; subsequently, there have been some new developments in quantum mechanics. The most important development is a definitive formulation of geometrical phases, introduced by M. V. Berry in 1983. The phase factors accompanying adiabatic changes are expressed in concise and elegant forms and have found universal applications in various fields of physics, thus giving a new viewpoint to quantum theory. We review here the physical consequences of these phases, which have in fact been used unconsciously in some cases already, by adding a supplement to the Japanese version of the text. (Here in the new English edition of Modern Quantum Mechanics we are providing a translation from Japanese of this supplement, prepared by Professor Akio Sakurai of Kyoto Sangyo University for the Japanese version of the book. The Editor deeply appreciates Professor Akio Sakurai’s guidance on an initial translation provided by his student, Yasunaga Suzuki, as a term paper for the graduate quantum mechanics course here at the University of Hawaii—Manoa.)
Basics

Parametric Hamiltonian, non degenerate ground state

\[ H(\xi) |\psi(\xi)\rangle = E(\xi) |\psi(\xi)\rangle \]

parameter \( \xi \): “slow variable”

\[ e^{-i\Delta \varphi_{12}} = \frac{\langle \psi(\xi_1) | \psi(\xi_2) \rangle}{|\langle \psi(\xi_1) | \psi(\xi_2) \rangle|} \]

\[ \Delta \varphi_{12} = - \text{Im} \log \langle \psi(\xi_1) | \psi(\xi_2) \rangle \]

\[ \gamma = \Delta \varphi_{12} + \Delta \varphi_{23} + \Delta \varphi_{34} + \Delta \varphi_{41} \]

\[ = - \text{Im} \log \langle \psi(\xi_1) | \psi(\xi_2) \rangle \langle \psi(\xi_2) | \psi(\xi_3) \rangle \langle \psi(\xi_3) | \psi(\xi_4) \rangle \langle \psi(\xi_4) | \psi(\xi_1) \rangle \]

Gauge-invariant!
Parametric Hamiltonian, non degenerate ground state

$$H(\xi)|\psi(\xi)\rangle = E(\xi)|\psi(\xi)\rangle$$

parameter $\xi$: “slow variable”

$$\gamma = \Delta \varphi_{12} + \Delta \varphi_{23} + \Delta \varphi_{34} + \Delta \varphi_{41}$$

$$= - \text{Im} \log \langle \psi(\xi_1)|\psi(\xi_2)\rangle\langle \psi(\xi_2)|\psi(\xi_3)\rangle\langle \psi(\xi_3)|\psi(\xi_4)\rangle\langle \psi(\xi_4)|\psi(\xi_1)\rangle$$

Gauge-invariant!
A smooth closed curve $C$ in $\xi$ space

$$e^{-i\Delta \varphi} = \frac{\langle \psi(\xi) | \psi(\xi + \Delta \xi) \rangle}{|\langle \psi(\xi) | \psi(\xi + \Delta \xi) \rangle|}$$

If we choose a **differentiable gauge**:

$$-i\Delta \varphi \simeq \langle \psi(\xi) | \nabla_\xi \psi(\xi) \rangle \cdot \Delta \xi$$

$$\gamma = \sum_{s=1}^{M} \Delta \varphi_{s,s+1} \rightarrow \oint_C d\varphi$$

$$d\varphi = A(\xi) \cdot d\xi = i \langle \psi(\xi) | \nabla_\xi \psi(\xi) \rangle \cdot d\xi$$

$d\varphi$  linear differential form, $i \langle \psi(\xi) | \nabla_\xi \psi(\xi) \rangle$  vector field
A smooth closed curve $C$ in $\xi$ space

$$e^{-i\Delta \varphi} = \frac{\langle \psi(\xi)|\psi(\xi+\Delta \xi) \rangle}{|\langle \psi(\xi)|\psi(\xi+\Delta \xi) \rangle|}$$

If we choose a **differentiable gauge**:

$$-i\Delta \varphi \approx \langle \psi(\xi)|\nabla_\xi \psi(\xi) \rangle \cdot \Delta \xi$$

$$\gamma = \sum_{s=1}^{M} \Delta \varphi_{s,s+1} \rightarrow \oint_{C} d\varphi$$

$$d\varphi = \mathcal{A}(\xi) \cdot d\xi = i \langle \psi(\xi)|\nabla_\xi \psi(\xi) \rangle \cdot d\xi$$

$d\varphi$ linear differential form, $i \langle \psi(\xi)|\nabla_\xi \psi(\xi) \rangle$ vector field
Berry connection & Berry curvature

- Domain $S$: $\xi \in S \subset \mathbb{R}^d$

- Berry connection
  $\mathcal{A}(\xi) = i \langle \psi(\xi) | \nabla_\xi \psi(\xi) \rangle$
  - real, nonconservative vector field
  - gauge-dependent
  - “geometrical” vector potential
  - a.k.a. “gauge potential”

- Berry curvature (\xi \in \mathbb{R}^3)
  $\Omega(\xi) = \nabla_\xi \times \mathcal{A}(\xi) = i \langle \nabla_\xi \psi(\xi) | \times | \nabla_\xi \psi(\xi) \rangle$
  - gauge-invariant (hence observable)
  - geometric analog of a magnetic field
  - a.k.a. “gauge field”
The Berry connection is real

\[ \langle \psi(\xi) | \psi(\xi) \rangle = 1 \quad \forall \xi \]

\[ \nabla_\xi \langle \psi(\xi) | \psi(\xi) \rangle = 0 \]

\[ = \langle \nabla_\xi \psi(\xi) | \psi(\xi) \rangle + \langle \psi(\xi) | \nabla_\xi \psi(\xi) \rangle \]

\[ = 2 \text{Re} \langle \psi(\xi) | \nabla_\xi \psi(\xi) \rangle \]

\[ \langle \psi(\xi) | \nabla_\xi \psi(\xi) \rangle \text{ purely imaginary} \]

\[ \mathcal{A}(\xi) = i \langle \psi(\xi) | \nabla_\xi \psi(\xi) \rangle \text{ real} \quad (1) \]

Last but not least:
What about time-reversal invariant systems?
The Berry connection is real

\[ \langle \psi(\xi) | \psi(\xi) \rangle = 1 \quad \forall \xi \]

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Last but not least:
What about time-reversal invariant systems?
Berry connection vs. perturbation theory

\[
|\psi_0(\xi + \Delta\xi)\rangle - |\psi_0(\xi)\rangle \\
\leq \sum_{n \neq 0}' |\psi_n(\xi)\rangle \frac{\langle \psi_n(\xi)| [ H(\xi + \Delta\xi) - H(\xi) ] |\psi_0(\xi)\rangle}{E_0(\xi) - E_n(\xi)}
\]

\[
|\partial_\alpha \psi_0(\xi)\rangle = \sum_{n \neq 0}' |\psi_n(\xi)\rangle \frac{\langle \psi_n(\xi)| \partial_\alpha H(\xi) |\psi_0(\xi)\rangle}{E_0(\xi) - E_n(\xi)}
\]

\[
A_\alpha(\xi) = i\langle \psi_0(\xi)| \partial_\alpha \psi_0(\xi)\rangle = 0
\]

“parallel transport” gauge
Berry connection vs. perturbation theory

\[ |\psi_0(\xi + \Delta\xi)\rangle - |\psi_0(\xi)\rangle \]

\[ \simeq \sum_{n \neq 0}' |\psi_n(\xi)\rangle \frac{\langle \psi_n(\xi) | [H(\xi + \Delta\xi) - H(\xi)] |\psi_0(\xi)\rangle}{E_0(\xi) - E_n(\xi)} \]

\[ |\partial_\alpha \psi_0(\xi)\rangle = \sum_{n \neq 0}' |\psi_n(\xi)\rangle \frac{\langle \psi_n(\xi) | \partial_\alpha H(\xi) |\psi_0(\xi)\rangle}{E_0(\xi) - E_n(\xi)} \]

\[ A_\alpha(\xi) = i\langle \psi_0(\xi) | \partial_\alpha \psi_0(\xi)\rangle = 0 \]

“parallel transport” gauge
Berry connection vs. perturbation theory

\[ \psi_0(\xi + \Delta \xi) - \psi_0(\xi) = \sum_{n \neq 0}' |\psi_n(\xi)\rangle \frac{\langle \psi_n(\xi) | [H(\xi + \Delta \xi) - H(\xi)] |\psi_0(\xi)\rangle}{E_0(\xi) - E_n(\xi)} \]

\[ \partial_\alpha \psi_0(\xi) = \sum_{n \neq 0}' |\psi_n(\xi)\rangle \frac{\langle \psi_n(\xi) | \partial_\alpha H(\xi) |\psi_0(\xi)\rangle}{E_0(\xi) - E_n(\xi)} \]

\[ \mathcal{A}_\alpha(\xi) = i \langle \psi_0(\xi) | \partial_\alpha \psi_0(\xi) \rangle = 0 \]

“parallel transport” gauge
Parallel transport

\[ |\Delta \psi_0(\xi)\rangle = \sum'_{n \neq 0} |\psi_n(\xi)\rangle \frac{\langle \psi_n(\xi)| [H(\xi + \Delta \xi) - H(\xi)]|\psi_0(\xi)\rangle}{E_0(\xi) - E_n(\xi)} \]

\[ |\psi_0(\xi + \Delta \xi)\rangle \approx |\psi_0(\xi)\rangle + |\Delta \psi_0(\xi)\rangle \]

\[ |\Delta \psi_0(\xi)\rangle \text{ orthogonal to } |\psi_0(\xi)\rangle \]

Differential Geometry:

Gaussian curvature of the spherical surface \( \Omega = 1/R^2 \)

\[ \int_\Sigma \Omega d\sigma = \text{angular mismatch} \]

Connection?
Berry connection vs. perturbation theory, better

\[ |\Delta\psi_0(\xi)\rangle = \sum_{n \neq 0} |\psi_n(\xi)\rangle \langle \psi_n(\xi)| \frac{[H(\xi + \Delta\xi) - H(\xi)]|\psi_0(\xi)\rangle}{E_0(\xi) - E_n(\xi)} \]

\[ |\psi_0(\xi + \Delta\xi)\rangle \simeq |\psi_0(\xi)\rangle + |\Delta\psi_0(\xi)\rangle \]

**Better:**

\[ |\psi_0(\xi + \Delta\xi)\rangle \rightarrow [ |\psi_0(\xi)\rangle + |\Delta\psi_0(\xi)\rangle ] e^{-i\Delta\varphi(\xi)} \]

\[ \simeq [1 - i\Delta\varphi(\xi)] |\psi_0(\xi)\rangle + |\Delta\psi_0(\xi)\rangle \]

\[ \mathcal{A}(\xi) \cdot d\xi = i \langle \psi_0(\xi)|\nabla_\xi \psi_0(\xi)\rangle \cdot d\xi \]

\[ = 0 + d\varphi \]
Berry connection vs. perturbation theory, better

\[ |\Delta \psi_0(\xi)\rangle = \sum_{n \neq 0} \langle \psi_n(\xi) | \left[ H(\xi + \Delta \xi) - H(\xi) \right] |\psi_0(\xi)\rangle \]

\[ |\psi_0(\xi + \Delta \xi)\rangle \simeq |\psi_0(\xi)\rangle + |\Delta \psi_0(\xi)\rangle \]

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Berry connection vs. perturbation theory, better

$$|\Delta \psi_0(\xi)\rangle = \sum_{n \neq 0}^\prime |\psi_n(\xi)\rangle \frac{\langle \psi_n(\xi) | [H(\xi + \Delta \xi) - H(\xi)] | \psi_0(\xi)\rangle}{E_0(\xi) - E_n(\xi)}$$

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$$|\psi_0(\xi + \Delta \xi)\rangle \rightarrow \left[ |\psi_0(\xi)\rangle + |\Delta \psi_0(\xi)\rangle \right] e^{-i\Delta \varphi(\xi)}$$

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$$\mathcal{A}(\xi) \cdot d\xi = i \langle \psi_0(\xi) | \nabla_{\xi} \psi_0(\xi) \rangle \cdot d\xi$$

$$= 0 + d\varphi$$
Berry curvature: perturbation theory is OK

The Berry curvature is **gauge invariant**

\[
\Omega(\xi) = \nabla_\xi \times A(\xi) \quad (\xi \in \mathbb{R}^3)
\]

\[
= i \sum_{n \neq 0} \frac{\langle \psi_0(\xi) | \nabla H(\xi) | \psi_n(\xi) \rangle \times \langle \psi_n(\xi) | \nabla H(\xi) | \psi_0(\xi) \rangle}{[E_0(\xi) - E_n(\xi)]^2}
\]

\[\Omega(\xi)\] singular at degeneracy points
The Berry curvature is \textbf{gauge invariant}.

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\( \Omega(\xi) \) singular at degeneracy points.
The Berry curvature is **gauge invariant**

\[ \Omega(\xi) = \nabla_{\xi} \times \mathcal{A}(\xi) \quad (\xi \in \mathbb{R}^3) \]

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\[ \frac{[E_0(\xi) - E_n(\xi)]^2}{[E_0(\xi) - E_n(\xi)]^2} \]

\[ \Omega(\xi) \text{ singular at degeneracy points} \]

Stokes’ theorem: \( C = \partial \Sigma \)

\[ \gamma = \oint_{\partial \Sigma} \mathcal{A}(\xi) \cdot d\xi = \int_{\Sigma} \Omega(\xi) \cdot n \ d\sigma \]

.......only if \( \Sigma \) is **simply connected**!
Berry phase

- Loop integral of the Berry connection on a closed path:

\[ \gamma = \oint_C A(\xi) \cdot d\xi \]

- Berry phase, gauge invariant modulo $2\pi$
- corresponds to *measurable* effects

**Main message** of Berry’s 1984 paper:

- In quantum mechanics, *any* gauge-invariant quantity is potentially a physical observable
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- corresponds to \textbf{measurable} effects

**Main message** of Berry’s 1984 paper:

- In quantum mechanics, \textbf{any} gauge-invariant quantity is potentially a physical observable
Coupling to “the rest of the Universe”

- $\gamma$ cannot be cast as the expectation value of any Hermitian operator: instead, it is a gauge-invariant phase of the wavefunction.

- The quantum system is not isolated: the parameter $\xi$ summarizes the effect of “the rest of the Universe”.

- **Slow variables**: $\xi$ (e.g., a nuclear coordinate).
  **Fast variables**: here, the electronic coordinates.

- For a genuinely isolated system, no Berry phase occurs and all observable effects are indeed expectation values of some operators.

- What about classical mechanics?
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- What about classical mechanics?
So far, everything \textit{time-independent}. Suppose instead that:

- The energy of $|\psi(\xi)\rangle$ is $E(\xi)$
- The parameter moves \textit{adiabatically} on the closed path in time $t$: $\xi \rightarrow \xi(t)$, with $\xi(T) = \xi(0)$

Then the state acquires a total phase factor $e^{i\gamma} e^{i\alpha(T)}$

- The phase $\gamma$ is independent of the details of motion: hence “geometric”
- The additional phase is the \textit{dynamical phase}, and does depend on the motion: $\alpha(T) = -\frac{1}{\hbar} \int_0^T dt \, E(\xi(t))$
Semantics: why “Geometric”? 

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Berry phase

- Loop integral of the Berry connection on a closed path:

\[ \gamma = \int_C A(\xi) \cdot d\xi \]

- Berry phase, gauge invariant only modulo \(2\pi\)
- Corresponds to measurable effects

- If \( C = \partial \Sigma \) is the boundary of \( \Sigma \), then (Stokes th.):

\[ \gamma = \int_{\partial \Sigma} A(\xi) \cdot d\xi = \int_{\Sigma} d\sigma \ \Omega(\xi) \cdot \hat{n} \]

- Requires \( \Sigma \) to be simply connected
- Requires \( A \) to be regular on \( \Sigma \)
- No longer arbitrary mod \(2\pi\)

- What about integrating the curvature on a \textbf{closed} surface?
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- What about integrating the curvature on a closed surface?
A simple example: Two level system

\[ H(\xi) = \xi \cdot \vec{\sigma} \quad \text{nondegenerate for } \xi \neq 0 \]
\[ = \xi (\sin \vartheta \cos \varphi \sigma_x + \sin \vartheta \sin \varphi \sigma_y + \cos \vartheta \sigma_z) \]

lowest eigenvalue \(-\xi\)

lowest eigenvector \( |\psi(\vartheta, \varphi)\rangle = \begin{pmatrix} \sin \frac{\vartheta}{2} e^{-i\varphi} \\ -\cos \frac{\vartheta}{2} \end{pmatrix} \)

\[ \mathcal{A}_\vartheta = i\langle \psi | \partial_\vartheta \psi \rangle = 0 \]
\[ \mathcal{A}_\varphi = i\langle \psi | \partial_\varphi \psi \rangle = \sin^2 \frac{\vartheta}{2} \]
\[ \Omega = \partial_\vartheta \mathcal{A}_\varphi - \partial_\varphi \mathcal{A}_\vartheta = \frac{1}{2} \sin \vartheta \]

- \( \Omega \) gauge invariant
- What about \( \mathcal{A} \)? Obstruction!
A simple example: Two level system

\[
H(\xi) = \xi \cdot \vec{\sigma} \quad \text{nondegenerate for } \xi \neq 0
\]

\[
= \xi \left( \sin \vartheta \cos \varphi \sigma_x + \sin \vartheta \sin \varphi \sigma_y + \cos \vartheta \sigma_z \right)
\]

lowest eigenvalue = $-\xi$

lowest eigenvector $|\psi(\vartheta, \varphi)\rangle = \begin{pmatrix} \sin \frac{\vartheta}{2} e^{-i\varphi} \\ -\cos \frac{\vartheta}{2} \end{pmatrix}$

\[
\mathcal{A}_\vartheta = i \langle \psi | \partial_\vartheta \psi \rangle = 0
\]

\[
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\]

\[
\Omega = \partial_\vartheta \mathcal{A}_\varphi - \partial_\varphi \mathcal{A}_\vartheta = \frac{1}{2} \sin \vartheta
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\[ = \xi (\sin \vartheta \cos \varphi \sigma_x + \sin \vartheta \sin \varphi \sigma_y + \cos \vartheta \sigma_z) \]

- lowest eigenvalue \(-\xi\)
- lowest eigenvector \(|\psi(\vartheta, \varphi)\rangle = \left( \begin{array}{c} \sin \frac{\vartheta}{2} e^{-i\varphi} \\ -\cos \frac{\vartheta}{2} \end{array} \right)\)

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- \(\Omega\) gauge invariant
- What about \(\mathcal{A}\)? Obstruction!
Integrating the Berry curvature

- Gauss-Bonnet-Chern theorem (1940):

\[ \frac{1}{2\pi} \int_{S^2} \Omega(\xi) \cdot \mathbf{n} \, d\sigma = \text{topological integer} \in \mathbb{Z} \]

- Integrating \( \Omega(\psi, \varphi) \) over \( [0, \pi] \times [0, 2\pi] \):

\[ \frac{1}{2\pi} \int d\psi d\varphi \frac{1}{2} \sin \psi = 1 \quad \text{Chern number } C_1 \]

- Measures the singularity at \( \xi = 0 \) (monopole)

- Berry phase on any closed curve \( C \) on the sphere:
Integrating the Berry curvature

- Gauss-Bonnet-Chern theorem (1940):
  \[ \frac{1}{2\pi} \int_{S^2} \Omega(\xi) \cdot n \, d\sigma = \text{topological integer} \in \mathbb{Z} \]
- Integrating $\Omega(\vartheta, \varphi)$ over $[0, \pi] \times [0, 2\pi]$:
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- Berry phase on any closed curve $C$ on the sphere:
  \[ \gamma \equiv \int_C A(\xi) \cdot d\xi \]
  \[ = \frac{1}{2} \times \text{(solid angle spanned)} \]
Integrating the Berry curvature

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  \[ \gamma \equiv \oint_C A(\xi) \cdot d\xi = \frac{1}{2} \times (\text{solid angle spanned}) \]
The sphere as the sum of two half spheres

\[ 2\pi C_1 = \int_{S^2} \mathbf{\Omega}(\xi) \cdot \mathbf{n} \, d\sigma \]

\[ = \int_{S_+} \mathbf{\Omega}(\xi) \cdot \mathbf{n} \, d\sigma + \int_{S_-} \mathbf{\Omega}(\xi) \cdot \mathbf{n} \, d\sigma \]

Stokes:

\[ \int_{S_{\pm}} \mathbf{\Omega}(\xi) \cdot \mathbf{n} \, d\sigma = \pm \oint_{C} \mathbf{A}_{\pm}(\xi) \cdot d\xi \]

\[ \int_{S^2} \mathbf{\Omega}(\xi) \cdot \mathbf{n} \, d\sigma = \oint_{C} \mathbf{A}_+(\xi) \cdot d\xi - \oint_{C} \mathbf{A}_-(\xi) \cdot d\xi \]

Gauge choice: \( \mathcal{A}_- (\xi) \) regular in the lower hemisphere; hence it has an \textbf{obstruction} in the upper hemisphere

\[ 2\pi C_1 = \oint_{S_+} \mathbf{\Omega}(\xi) \cdot \mathbf{n} \, d\sigma - \oint_{C} \mathbf{A}_-(\xi) \cdot d\xi \]
The sphere as the sum of two half spheres

\[ 2\pi C_1 = \int_{S^2} \Omega(\xi) \cdot n \, d\sigma \]

\[ = \int_{S_+} \Omega(\xi) \cdot n \, d\sigma + \int_{S_-} \Omega(\xi) \cdot n \, d\sigma \]

Stokes:

\[ \int_{S_{\pm}} \Omega(\xi) \cdot n \, d\sigma = \pm \oint_{C} A_{\pm}(\xi) \cdot d\xi \]

\[ \int_{S^2} \Omega(\xi) \cdot n \, d\sigma = \oint_{C} A_{+}(\xi) \cdot d\xi - \oint_{C} A_{-}(\xi) \cdot d\xi \]

Gauge choice: \( A_{-}(\xi) \) regular in the lower hemisphere:

hence it has an \textbf{obstruction} in the upper hemisphere

\[ 2\pi C_1 = \int_{S_+} \Omega(\xi) \cdot n \, d\sigma - \oint_{C} A_{-}(\xi) \cdot d\xi \]
Bloch orbitals (noninteracting electrons in this talk)

- Lattice-periodical Hamiltonian (no macroscopic B field); 2d, single band, spinless electrons

\[ H \ket{\psi_k} = \varepsilon_k \ket{\psi_k} \]
\[ H_k \ket{u_k} = \varepsilon_k \ket{u_k} \quad \ket{u_k} = e^{-i k \cdot r} \ket{\psi_k} \quad H_k = e^{-i k \cdot r} H e^{i k \cdot r} \]

- Berry connection and curvature \((\xi \rightarrow k)\):

\[ A(k) = i \langle u_k | \nabla_k u_k \rangle \]
\[ \Omega(k) = i \langle \nabla_k u_k \rangle \times \ket{\nabla_k u_k} = -2 \text{ Im} \langle \partial_{k_x} u_k \partial_{k_y} u_k \rangle \]

- BZ (or reciprocal cell) is a closed surface: 2d torus

Topological invariant:

\[ C_1 = \frac{1}{2\pi} \int_{\text{BZ}} d\mathbf{k} \ \Omega(k) \quad \text{Chern number} \]
Bloch orbitals (noninteracting electrons in this talk)

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Chern number
Computing the Chern number

Discretized reciprocal cell
Computing the Chern number

Discretized reciprocal cell

Periodic gauge choice: where is the obstruction?
Computing the Chern number

Discretized reciprocal cell

Curvature $\equiv$ Berry phase per unit (reciprocal) area
Berry phase on a small square:

$$\gamma = -\text{Im} \log \langle u_{k_1} | u_{k_2} \rangle \langle u_{k_2} | u_{k_1} \rangle \langle u_{k_3} | u_{k_2} \rangle \langle u_{k_2} | u_{k_3} \rangle \langle u_{k_4} | u_{k_2} \rangle \langle u_{k_2} | u_{k_4} \rangle$$
Computing the Chern number

Discretized reciprocal cell

Curvature ≡ Berry phase per unit (reciprocal) area
Berry phase on a small square:

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Which branch of Im log?
Computing the Chern number

Discretized reciprocal cell

NonAbelian (many-band):

\[ \gamma = -\text{Im} \log \det S(k_1, k_2)S(k_2, k_3)S(k_3, k_4)S(k_4, k_1) \]

\[ S_{nn'}(k_s, k_{s'}) = \langle u_{nk_s} | u_{nk_{s'}} \rangle \]
Outline

1. Appendix: Metric and curvature
The simplest geometrical property: Distance

Two state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ in the same Hilbert space

$$D^2_{12} = - \log |\langle \psi_1 | \psi_2 \rangle|^2$$

- $D^2_{12} = 0$ if the two quantum states coincide apart for an irrelevant phase: **gauge-invariant**
- $D^2_{12} = \infty$ if the two states are orthogonal
A second geometrical property: Connection

\[ D_{12}^2 = - \log |\langle \psi_1 | \psi_2 \rangle|^2 = - \log \langle \psi_1 | \psi_2 \rangle - \log \langle \psi_2 | \psi_1 \rangle \]

- The two terms are **not** gauge-invariant
- Each of the two terms is a complex number
- What is the meaning of \( \text{Im} \log \langle \psi_1 | \psi_2 \rangle \)?

\[ \langle \psi_1 | \psi_2 \rangle = |\langle \psi_1 | \psi_2 \rangle| e^{i \varphi_{12}} \]

\[ - \text{Im} \log \langle \psi_1 | \psi_2 \rangle = \varphi_{12}, \quad \varphi_{21} = - \varphi_{12} \]

- The connection fixes the **phase difference**
- The connection is **arbitrary**
- Given that it is arbitrary, **why bother?**
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- The connection is arbitrary
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Differential quantities in quantum geometry

The state vector $|\psi_\kappa\rangle$ depends on the continuous parameter $\kappa$.

- Quantum metric $g_{\alpha\beta}$:
  \[ d \, D^2 = D^2_{\kappa, \kappa + d\kappa} = g_{\alpha\beta} d\kappa^\alpha d\kappa^\beta \]

- Berry connection $A_\alpha$:
  \[ d \varphi = A_\alpha d\kappa_\alpha \]

- Berry curvature $\Omega_{\alpha\beta} = \partial_{\kappa_\alpha} A_\beta - \partial_{\kappa_\beta} A_\alpha$
  \[ d \times d \varphi = \Omega_{\alpha\beta} d\kappa_\alpha d\kappa_\beta \]

All of the above depend on the state vector only.
Differential quantities in quantum geometry

The state vector $|\psi_\kappa\rangle$ depends on the continuous parameter $\kappa$

- Quantum metric:
  \[ dD^2 = D^2_{\kappa,\kappa + d\kappa} = g_{\alpha\beta} d\kappa_\alpha d\kappa_\beta \quad \text{2-form} \]

- Berry connection:
  \[ d\varphi = A_\alpha d\kappa_\alpha \quad \text{1-form} \]

- Berry curvature
  \[ d \times d\varphi = \Omega_{\alpha\beta} d\kappa_\alpha d\kappa_\beta \quad \text{2-form} \]

- All of the above depend on the state vector only
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  \[
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- Berry curvature
  \[
  d \times d\varphi = \Omega_{\alpha\beta} d\kappa_\alpha d\kappa_\beta
  \]

- All of the above depend on the state vector only
A more general geometrical quantity

- Beside the state vectors, even the Hamiltonian is involved:

\[ H |\psi_0\rangle = E_0 |\psi_0\rangle \]
\[ G = \langle \psi_\kappa | (H - E_0) |\psi_\kappa\rangle \]

- \( G \) vanishes when \( \psi_\kappa = \psi_0 \)

- \( G \) is invariant by translation of the energy zero

- Differential of \( G \)
  (when \( |\psi_\kappa\rangle \) is varied in a neighborhood of \( |\psi_0\rangle \))

\[ d G = \langle \psi_{d\kappa} | (H - E_0) |\psi_{d\kappa}\rangle \]
\[ = \langle \partial_{\kappa\alpha} \psi | (H - E_0) |\partial_{\kappa\beta} \psi \rangle d_{\kappa\alpha} d_{\kappa\beta} \]
A more general geometrical quantity

Beside the state vectors, even the Hamiltonian is involved:

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\[ = \langle \partial_\kappa^\alpha \psi | (H - E_0) |\partial_\kappa^\beta \psi \rangle d\kappa_\alpha d\kappa_\beta \quad 2\text{-form} \]