Integer Quantum Hall Effect

(Dawn of topology in electronic structure)

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Gaussian (a.k.a. CGS) units

- Permittivity of free space $\varepsilon_0 = \frac{1}{4\pi}$
- Permeability of free space $\mu_0 = 4\pi$
- In vacuo $\mathbf{D} \equiv \mathbf{E}$ and $\mathbf{H} \equiv \mathbf{B}$
- All fields have the same dimensions

- Newtonian & Hamiltonian mechanics:

$$M\frac{d\mathbf{v}}{dt} = \mathbf{f} = Q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

$$\mathcal{H} = \frac{1}{2M}\left(\mathbf{p} - \frac{Q}{c}\mathbf{A}(\mathbf{r})\right)^2 + Q\Phi(\mathbf{r})$$

Atomic Gaussian units

$$\mathcal{H} = \frac{1}{2M}\left(\mathbf{p} - \frac{1}{c}\mathbf{A}(\mathbf{r})\right)^2 + Q\Phi(\mathbf{r})$$

- Schrödinger Hamiltonian for the electron

$$\mathcal{H} = \frac{1}{2m_\mathrm{e}}\left(-i\hbar\nabla + \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2 - e\Phi(\mathbf{r})$$

- $m_\mathrm{e} = 1, \quad \hbar = 1, \quad e = 1, \qquad (c = 137)$
  1 a.u. of energy = 1 hartree = 2 rydberg = 27.21 eV

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Outline

Figure from Kittel ISSP, Ch. 6

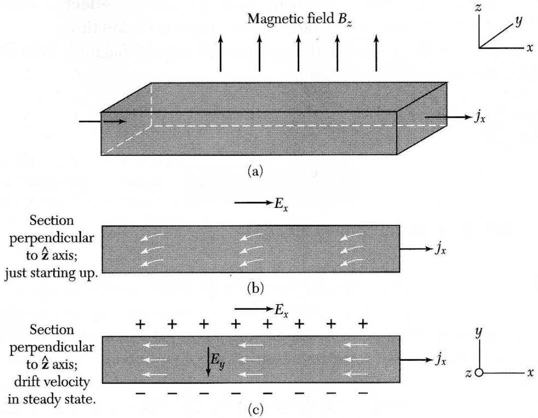
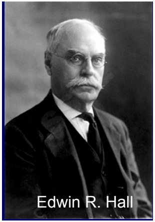
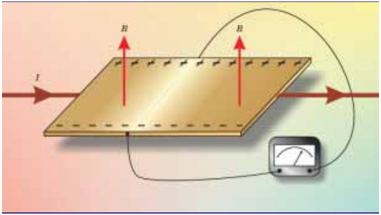


Figure 14 The standard geometry for the Hall effect: a rod-shaped specimen of rectangular cross-section is placed in a magnetic field B_z, as in (a). An electric field E_x applied across the end electrodes causes an electric current density j_x to flow down the rod. The drift velocity of the negatively-charged electrons immediately after the electric field is applied as shown in (b). The deflection in the $-y$ direction is caused by the magnetic field. Electrons accumulate on one face of the rod and a positive ion excess is established on the opposite face until, as in (c), the transverse electric field (Hall field) just cancels the Lorentz force due to the magnetic field.

Hall effect (1879)





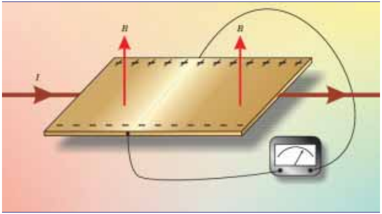
From Kittel ISSP (carriers of mass m and charge $-e$)

$$m\left(\frac{d\mathbf{v}}{dt} + \frac{1}{\tau}\mathbf{v}\right) = -e\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

Steady-state: $$\frac{d\mathbf{v}}{dt} = 0$$

Drude-Zener theory

$$\mathbf{v} = -\frac{e\tau}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$$



In 2d, set $E_y = 0$; cyclotron frequency $\omega_c = \frac{eB}{mc}$

$$\begin{aligned} v_x &= -\frac{e\tau}{m}E_x - \omega_c \tau v_x \\ v_y &= \omega_c \tau v_x \end{aligned}$$

Hall conductivity

Current $\mathbf{j} = -ne\,\mathbf{v}$ (n carrier density)

$$\begin{aligned} j_x &= \frac{ne^2\tau}{m} E_x - \omega_c \tau j_x \\ j_y &= \omega_c \tau j_x \end{aligned}$$

In zero **B** field

$$j_x = \sigma_0 E_x, \qquad \sigma_0 = \frac{ne^2\tau}{m}$$

In a **B** field

$$\begin{aligned} j_x &= \frac{\sigma_0}{1 + (\omega_c\tau)^2} E_x = \sigma_{xx} E_x \\ j_y &= \frac{\omega_c \tau \sigma_0}{1 + (\omega_c\tau)^2} E_x = \sigma_{yx} E_x \end{aligned}$$

Conductivity vs. resistivity (classical & quantum)

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & -\sigma_{yx} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\overleftrightarrow{\rho} = (\overleftrightarrow{\sigma})^{-1}$$

$$\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{yx}^2}, \qquad \rho_{xy} = \frac{\sigma_{yx}}{\sigma_{xx}^2 + \sigma_{yx}^2}$$

- At $\mathbf{B} = 0$ $\qquad \rho_{xx} = 1/\sigma_{xx}$
- In the nondissipative regime ($\mathbf{j} \cdot \mathbf{E} = 0$)

$$\sigma_{xx} = 0 \qquad \textbf{and} \qquad \rho_{xx} = 0$$

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Nondissipative limit ($\tau \to \infty$, classical Drude-Zener)

$$\sigma_0 = \frac{ne^2\tau}{m} \qquad \sigma_{xx} = \frac{\sigma_0}{1+(\omega_c\tau)^2} \qquad \sigma_{yx} = \frac{\omega_c\tau\sigma_0}{1+(\omega_c\tau)^2}$$

- At $\mathbf{B} = 0$ $\quad \sigma_{xx} = \sigma_0$ $\quad$ diverges
- At $\mathbf{B} \neq 0$ $\quad$ for $\quad \tau \gg 1/\omega_c$

$$\begin{aligned} \sigma_{xx} &= 0, \qquad \rho_{xx} = 0 \qquad \text{(longitudinal resistivity)} \\ \rho_{xy} &= 1/\sigma_{yx} = \frac{m\omega_c}{ne^2} = \frac{m}{ne^2}\frac{eB}{mc} \\ &= \frac{1}{nec}B \qquad \text{(Hall resistivity)} \end{aligned}$$

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Multiplying and dividing by h

- In 2d resistance/resistivity and conductance/conductivity have **the same dimensions**: do they coincide?
- $n = N/A$ (number of carrriers per unit area)

$$\rho_{xy} = \frac{1}{nec}B = \frac{AB}{Nec} = \frac{\Phi}{Nec} = \frac{1}{\nu}\frac{h}{e^2}$$

- Φ magnetic flux through area A
  $h/e^2 \simeq 25813\ \Omega$ (natural resistance unit)
  ν dimensionless

$$\nu = \frac{N\Phi_0}{\Phi} \quad \text{filling factor,} \qquad \Phi_0 = \frac{hc}{e} \quad \text{flux quantum}$$

$\nu =$ (number of electrons)/(number of flux quanta)

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Experiment (von Klitzing 1980, Nobel prize 1985)

VOLUME 45, NUMBER 6 PHYSICAL REVIEW LETTERS 11 AUGUST 1980

New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance

K. v. Klitzing
Physikalisches Institut der Universität Würzburg, D-8700 Würzburg, Federal Republic of Germany, and Hochfeld-Magnetlabor des Max-Planck-Instituts für Festkörperforschung, F-38042 Grenoble, France

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M. Pepper
Cavendish Laboratory, Cambridge CB3 0HE, United Kingdom
(Received 30 May 1980)

Measurements of the Hall voltage of a two-dimensional electron gas, realized with a silicon metal-oxide-semiconductor field-effect transistor, show that the Hall resistance at particular, experimentally well-defined surface carrier concentrations has fixed values which depend only on the fine-structure constant and speed of light, and is insensitive to the geometry of the device. Preliminary data are reported.

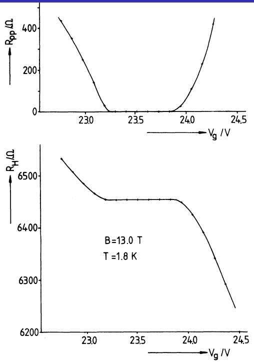

FIG. 2. Hall resistance R_H, and device resistance, R_{pp}, between the potential probes as a function of the gate voltage V_g in a region of gate voltage corresponding to a fully occupied, lowest ($n = 0$) Landau level. The plateau in R_H has a value of 6453.3 ± 0.1 Ω. The geometry of the device was $L = 400$ μm, $W = 50$ μm, and $L_{pp} = 130$ μm; $B = 13$ T.

$h/e^2 = 25812.807557(18)\ \Omega \quad = 1$ klitzing

Since 1990 a new metrology standard

In the original experiment (MOSFET): $\nu = 4$

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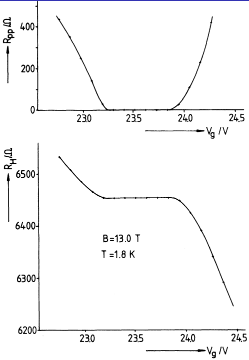

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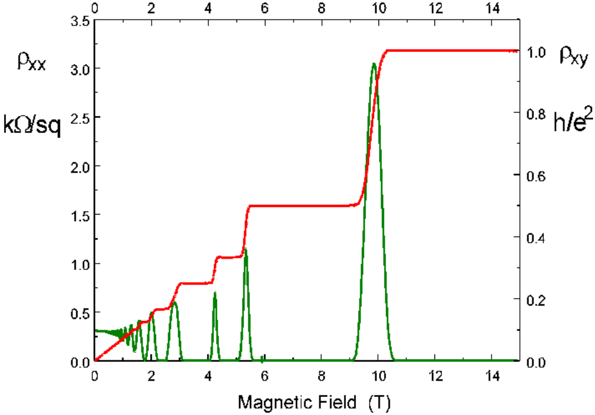
More recent experiments



GaAs-GaAlAs heterojunction, at 30mK

Outline

Hamiltonian in **B** field (flat substrate potential)

N noninteracting (& spin-polarized) electrons in zero potential:

$$\hat{H} = \frac{1}{2m_\mathrm{e}} \sum_{i=1}^{N} \left[\mathbf{p}_i + \frac{e}{c}\mathbf{A}(\mathbf{r}_i)\right]^2$$

- Gaussian units
- m_e electron mass
- $-e$ electron charge
- $\frac{1}{m_\mathrm{e}}\left(\mathbf{p}_i + \frac{e}{c}\mathbf{A}(\mathbf{r}_i)\right)$ velocity
- $\mathbf{p}_i = -i\hbar\nabla_i$ canonical momentum
- $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{r})$

Landau gauge

Everything in **2d**; **B** uniform, along z.

$$A_x = 0, \qquad A_y = Bx$$

For each electron the Hamiltonian is

$$H(x,y) = \frac{\hbar^2}{2m_\mathrm{e}}\left[-\frac{\partial^2}{\partial x^2} + \left(-i\frac{\partial}{\partial y} + \frac{e}{\hbar c}Bx\right)^2\right]$$

Landau ansatz $\psi_k(x,y) = \mathrm{e}^{iky}\varphi_k(x)$

$$-\frac{\hbar^2}{2m_\mathrm{e}}\mathrm{e}^{iky}\varphi_k''(x) + \frac{\hbar^2}{2m_\mathrm{e}}\left(k + \frac{eB}{\hbar c}x\right)^2 \mathrm{e}^{iky}\varphi_k(x) = \varepsilon_k\, \mathrm{e}^{iky}\varphi_k(x).$$

Harmonic oscillator in 1d

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$$-\frac{\hbar^2}{2m_e}\varphi_k''(x) + \frac{1}{2}m_e\left(\frac{eB}{m_e c}\right)^2\left(x + \frac{\hbar c}{eB}k\right)^2 \varphi_k(x) = \varepsilon_k\, \varphi_k(x)$$

Harmonic oscillator

- Center in $x_k = -\frac{\hbar c}{eB}k = -\ell^2 k$
  $\ell = (\hbar c/eB)^{1/2}$ "magnetic length" (diverges for $B \to 0$)
- Frequency $\omega_c = \frac{eB}{m_e c}$ cyclotron frequency
  (classical, Gaussian units)

Landau oscillator

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Eigenvalues and eigenvectors

- Spectrum **independent** of k: $\varepsilon_n = (n + \frac{1}{2})\omega_c$
- Ground-state orbitals (LLL):

$$\psi_k(x,y) = e^{iky}\varphi_k(x) = e^{iky}\chi(x + \ell^2 k)$$

$$\chi(x) = \left(\frac{1}{\pi\ell^2}\right)^{1/4} e^{-x^2/(2\ell^2)}$$

- Infinite degeneracy: one orbital for each k
- Electron confined in a vertical strip centered at $\ell^2 k$
- What about the current?
- Any unitary transformation of the LLL orbitals is an eigenfunction

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Counting the states (discretize k)

$$\psi_k(x,y) = \mathrm{e}^{iky}\chi(x-\ell^2 k) \qquad \chi(x) = \left(\frac{1}{\pi\ell^2}\right)^{1/4} \mathrm{e}^{-x^2/(2\ell^2)}$$

- Periodic boundary conditions in y: $k_{i+1} - k_i = \frac{2\pi}{L}$
- Horizontal distance between neighboring orbitals: $\frac{2\pi\ell^2}{L}$
- Area covered by one state: $2\pi\ell^2$
  Number of states in each LL: $\mathcal{N} = \frac{A}{2\pi\ell^2}$
- Magnetic flux:
  $\Phi = AB = \mathcal{N}2\pi\ell^2 B = \mathcal{N}\frac{2\pi\hbar c}{e} = \mathcal{N}\frac{hc}{e} = \mathcal{N}\Phi_0$
- Flux quantum: $\Phi_0 = \frac{hc}{e}$ ($\Phi_0 = \frac{h}{e}$ in SI units)
- Φ_0 a **universal constant**

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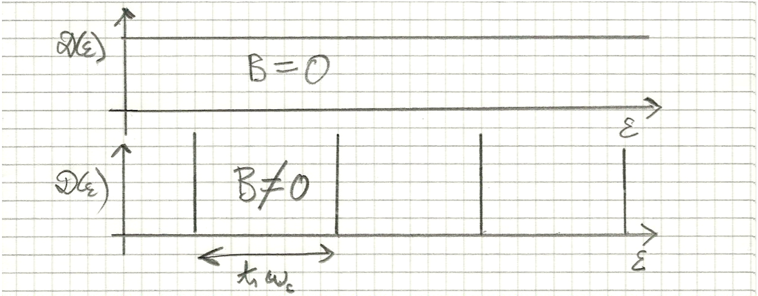
Density of states



- At $B = 0$: $\mathcal{D}(\varepsilon)$ = constant = $\frac{2\pi m_e A}{h^2}$
- At $B \neq 0$: Φ/Φ_0 states in each LL

$$\mathcal{D}(\varepsilon) = \frac{\Phi}{\Phi_0} \sum_{n=1}^{\infty} \delta\left(\varepsilon - (n + \frac{1}{2})\hbar\omega_c\right)$$

maximum filling for each LL is $\nu = 1$.

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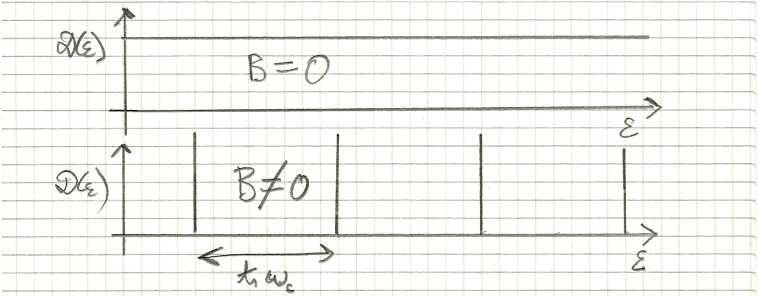

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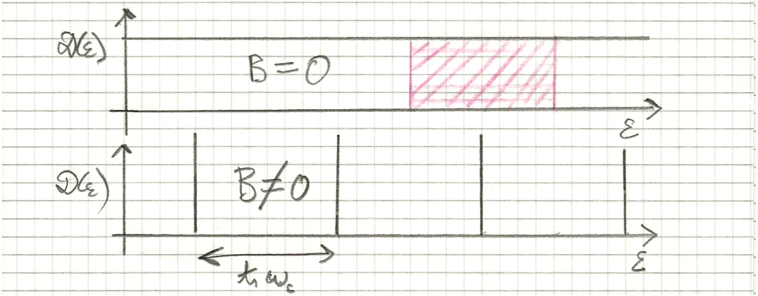
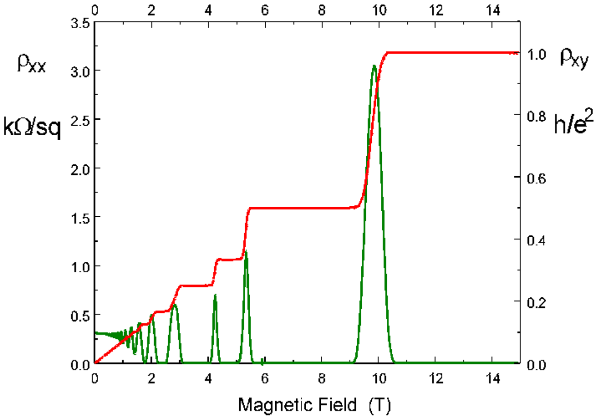


- How many states in the hatched region?

$$\int_{\varepsilon}^{\varepsilon+\hbar\omega_c} d\varepsilon' \, \mathcal{D}(\varepsilon') = \hbar\omega_c \frac{2\pi m_e A}{h^2} = \frac{\Phi}{\Phi_0}$$

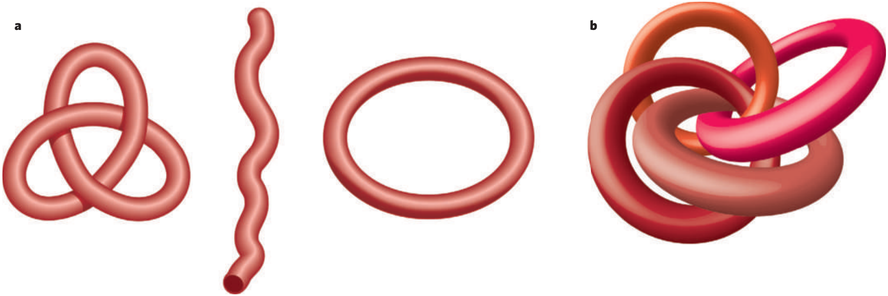
Outline

What the experiment shows



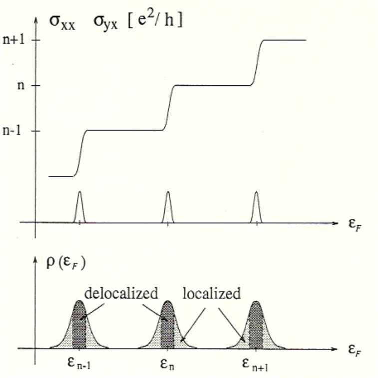
In modern jargon: The plateaus are “topologically protected”

Wavefunction “knotted” or “twisted”



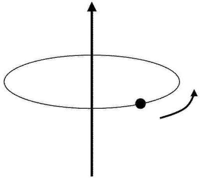
- Knotted in reciprocal space in nontrivial ways
- The famous TKNN paper:
  D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
- Integer numbers are very “robust”

Role of disorder



Current carried by delocalized states only

Varying the "inaccessible flux"

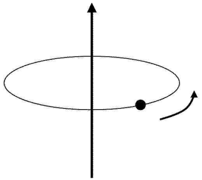


- In a flat potential: $\varepsilon_n(\varphi) = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2\left(n + \frac{\varphi}{\Phi_0}\right)^2$
- Hellmann-Feynman theorem (in **any** potential):

$$v = \frac{1}{\hbar}\frac{\partial H}{\partial \kappa} \qquad \langle \psi_n | v | \psi_n \rangle = \frac{1}{\hbar}\frac{d\epsilon_n(\kappa)}{d\kappa}$$

- Next: N noninteracting electrons in an arbitrary potential

Topological robustness of the current

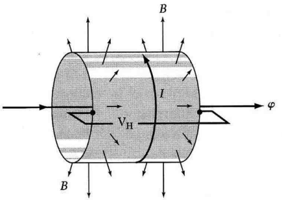


$$U = \sum_{n \in \text{occupied}} \epsilon_n \qquad I = -\frac{1}{c}\frac{\partial U}{\partial \varphi}$$

- **Independent** of the substrate potential
  Independent on the number N of current carrying states
- Variation of a full flux quantum:

$$\Delta U = U(\varphi + \Phi_0) - U(\varphi) = -\frac{\Phi_0 I}{c}$$

Laughlin's Gedankenexperiment (1981)



- The insertion of a flux quantum Φ_0 maps the system into itself: how can the energy vary?
- Answer: an integer number ν of electrons is transferred from one edge to the other
- If the edges are kept at voltage V_y, then

$$\nu e V_y = \Delta U = \frac{\Phi_0 I_x}{c}; \qquad R_{\mathrm{H}} = V_y / I_x = \frac{\phi_0}{\nu c e} = \frac{1}{\nu}\frac{h}{e^2}$$

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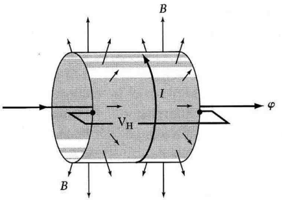

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