Semiclassical theory of conduction in metals - The Boltzmann equation

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Condensed Matter Physics I

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(reference: Ashcroft & Mermin, Solid State Physics, Ch 13 & 16)

1 Introduction - Electronic transport - generalities

Up to here, we have applied the semiclassical model of electron dynamics to the cases of :
- static electric fields
- uniform and static magnetic fields
- uniform and static electric and magnetic crossed fields

From now on we will treat a **more general case** (presence of $\mathbf{E}$, $\mathbf{H}$, ∇T, functions of $\mathbf{r}$ and t) but always considering:

- Independent electrons
- semiclassical motion between collisions
- no interband transition (conservation of band index n)
- no spin change (conservation of spin).

2 Sources of electronic scattering (collisions)

Ch 16 §I

- Perfect periodic crystal $\Longrightarrow$ NO COLLISIONS
- INDEPENDENT ELECTRON PICTURE (here!): two kinds of collisions:
  1. point defects, impurities, vacancies
  2. thermal effects $\Longrightarrow$ vibrations (small!) of electrons around their equilibrium positions (amplitude of vibrations depending on T: important scattering source in DC and main responsible of the T dependence of conductivity around R.T.; as $T \to 0$K, defects dominate)
- BEYOND INDEPENDENT ELECTRON APPROXIMATION:
  e^-–e^- scattering (e^-–e^- interactions) that are however:
  $<<$ T-dependent lattice vibrations effects at hight T;
  $<<$ impurity effects at low T

3 Non-equilibrium distribution function

A&M Ch. 13 §Introduction p 244 + Ch. 16 §IV p 319 + CH. 16 §I p 315

3.1 Generalities

We consider a **non equilibrium distribution function** $g(\mathbf{r}, \mathbf{k}, t)$ **in phase space** (in general, due to *applied fields* or *temperature gradients*), Is the occupancy of state $\mathbf{k}$ at position $\mathbf{r}$ and time dt.
External forces act to drive the distribution function **away** from equilibrium.
The limiting case at equilibrium, with NO applied fields, NO T gradients), corresponds to:

$$g^0(\mathbf{k}) = f(\mathcal{E}(\mathbf{k})) = \frac{1}{e^{(\mathcal{E}(\mathbf{k})-\mu)/k_B T} + 1} \tag{1}$$

AIM: derive a **closed** expression for $g(\mathbf{r}, \mathbf{k}, t)$ (when possible!) using:
- assumption of semiclassical equations of motion between collisions
- simple treatment of collisions

GENERALIZATION (the simplest possible!) OF THE RELAXATION TIME:
We continue to assume that an e^- experiences a collision in a time interval $dt$with probability dt/τ, but τ is in general $\tau(\mathbf{r},\mathbf{k})$.
This sounds reasonable, since –even in the independent electron approximation– collisions are not simply random and uncorrelated. They depend on the distribution of the other e^-, at least as far as occupation of levels is concerned with!

3.2 Differential equation for $g(\mathbf{r}, \mathbf{k}, t)$

We construct in general $g(\mathbf{r}, \mathbf{k}, t)$ at time t from its value at time $t' = t - dt$
a) first assuming NO collision during the infinitesimal time interval dt
b) consider $\mathbf{r}$ and $\mathbf{k}$ evolving according to the semiclassical equations of motion:

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{k}) \tag{2}$$

$$\hbar\dot{\mathbf{k}} = -e(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}) = \mathbf{F}(\mathbf{r}, \mathbf{k}). \tag{3}$$

We consider explicitly the time evolution of $\mathbf{r}$ and $\mathbf{k}$ from t' to t (linear expansion in dt):

$$t' = t - dt \qquad \longrightarrow t \tag{4}$$

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}(\mathbf{k})dt \qquad \longrightarrow \mathbf{r} \tag{5}$$

$$\mathbf{k}' = \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt \qquad \longrightarrow \mathbf{k} \tag{6}$$

The number of e^- that occupy the volume of phase space $\Delta\mathbf{r}\Delta\mathbf{k}$ centered in $\mathbf{r}$ and $\mathbf{k}$ at the time t is:

$$\frac{\Delta\mathbf{r}\Delta\mathbf{k}}{8\pi^3} g(\mathbf{r}, \mathbf{k}, t). \tag{7}$$

(we do not consider spin degeneracy, since we focus on a precise spin channel and we remain in that one). Analogously, the number of e^- occupying the volume of phase space

$\Delta\mathbf{r}'\Delta\mathbf{k}'$ centered in $\mathbf{r}$' and $\mathbf{k}$' at the time t' is:

$$\frac{\Delta\mathbf{r}'\Delta\mathbf{k}'}{8\pi^3} g(\mathbf{r}', \mathbf{k}', t') = \frac{\Delta\mathbf{r}'\Delta\mathbf{k}'}{8\pi^3} g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right). \tag{8}$$

If there are **no collisions (no scattering)**, the trajectory of the e^- in phase space is such that all the e^- that are in the volume of phase space $\Delta\mathbf{r}'\Delta\mathbf{k}'$ centered in $\mathbf{r}$' and $\mathbf{k}$' at the time t', are also in the volume of phase space $\Delta\mathbf{r}\Delta\mathbf{k}$ centered in $\mathbf{r}$ and $\mathbf{k}$ at the time t, therefore:

$$\frac{\Delta\mathbf{r}\Delta\mathbf{k}}{8\pi^3} g(\mathbf{r}, \mathbf{k}, t) = \frac{\Delta\mathbf{r}'\Delta\mathbf{k}'}{8\pi^3} g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right). \tag{9}$$

For the Liouville theorem:

$$\frac{\Delta\mathbf{r}\Delta\mathbf{k}}{8\pi^3} = \frac{\Delta\mathbf{r}'\Delta\mathbf{k}'}{8\pi^3} \tag{10}$$

therefore in absence of scattering:

$$g(\mathbf{r}, \mathbf{k}, t) - g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right) = 0. \tag{11}$$

Finally, we consider the expansion of $g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right)$ in terms of $g(\mathbf{r}, \mathbf{k}, t)$ up to the linear term in dt and we get:

$$g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right) = g(\mathbf{r}, \mathbf{k}, t) - \frac{\partial}{\partial\mathbf{r}} g \cdot \mathbf{v}(\mathbf{k})dt - -\frac{\partial}{\partial\mathbf{k}} g \cdot \frac{\mathbf{F}}{\hbar}dt - -\frac{\partial}{\partial t} g\ dt \tag{12}$$

therefore in absence of scattering we get:

$$\frac{\partial}{\partial\mathbf{r}} g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial\mathbf{k}} g \cdot \frac{\mathbf{F}}{\hbar} + \frac{\partial}{\partial t} g = 0 \tag{13}$$

Introducing collisions, i.e., generalizing **in presence of scattering**, the right hand side of the previous equation is no longer zero, but it is the contribution due to scattered electrons:

$$\boxed{\underbrace{\frac{\partial}{\partial\mathbf{r}} g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial\mathbf{k}} g \cdot \frac{\mathbf{F}}{\hbar} + \frac{\partial}{\partial t} g} = \underbrace{\left(\frac{\partial g}{\partial t}\right)_{coll}}} \tag{14}$$

where the left side is the *DRIFT term* and the right side is the *COLLISION term.*

This is the celebrated BOLTZMANN EQUATION.

If we specify the forces and the collision term, we have an initial value problem to determine $g(\mathbf{r}, \mathbf{k}, t)$. The difficult part now is related to the explicit knowledge of the collision term, that in general could be very complicate.
The simplest possible approximation is the *relaxation time approximation* to replace the right hand side term with something much more simple, but this will come later on.

3.3 Change of $g(\mathbf{r}, \mathbf{k}, t)$ due to collisions

A&M CH. 16 §IV

We assume that the collisions are very well **localized in space and time**, so that those occurring to electrons in $\mathbf{r}$ at time t are totally determined by the properties of the system in the neighborhoods of $\mathbf{r}$ and close to time $t \Longrightarrow$ for simplicity, we drop the explicit dependence of $g(\mathbf{r}, \mathbf{k}, t)$ on $\mathbf{r}$ and t in what follows.

We consider therefore collisions that change istantaneously the crystal momentum from a volume $\Delta\mathbf{k}$ centered in $\mathbf{k}$ to a volume $\Delta\mathbf{k}$' centered in $\mathbf{k}$'. We distinguish scattering events that **increase** the occupancy of electronic states at $\mathbf{k}$ and those that **decrease** it:

$$\left(\frac{\partial g}{\partial t}\right)_{coll}^{IN} > 0, \qquad \left(\frac{\partial g}{\partial t}\right)_{coll}^{OUT} < 0. \tag{15}$$

We do not need to specify which is the scattering mechanism; it will be described in any case by some *scattering matrix* $W_{\mathbf{k}\mathbf{k}'}$ for one electron suffering a scattering from a state $\mathbf{k}$ to $\mathbf{k}$'.

OUT: The e^- that are scattered OUT from the volume $\Delta\mathbf{k}$ centered in $\mathbf{k}$ will go somewhere else in phase space, in volumes $\Delta\mathbf{k}$' centered in all the possible $\mathbf{k}$', provided that the states are available (not already occupied, because the Pauli principle must be satisfied). Therefore, the contribution of such scattering events to the variation of $g(\mathbf{r}, \mathbf{k}, t)$ is:

$$\left(\frac{\partial g}{\partial t}\right)_{coll}^{OUT} = -g(\mathbf{k}) \int \frac{\Delta\mathbf{k}'}{8\pi^3} W_{\mathbf{k}\mathbf{k}'} \left(1 - g(\mathbf{k}')\right) \tag{16}$$

where we have also introduced a weighting factor $g(\mathbf{k})$ which is the occupancy of the states in the volume $\Delta\mathbf{k}$ centered in $\mathbf{k}$ and the sign $-$ since it is a reduction in the occupancy.

IN: The e^- that are scattered IN the volume $\Delta\mathbf{k}$ centered in $\mathbf{k}$ come from somewhere else in phase space, from volumes $\Delta\mathbf{k}$' centered in all the possible $\mathbf{k}$', provided that those states are filled. Therefore, the contribution of such scattering events to the variation of $g(\mathbf{r}, \mathbf{k}, t)$ is:

$$\left(\frac{\partial g}{\partial t}\right)_{coll}^{IN} = \left(1 - g(\mathbf{k})\right) \int \frac{\Delta\mathbf{k}'}{8\pi^3} W_{\mathbf{k}'\mathbf{k}} g(\mathbf{k}') \tag{17}$$

where we have also introduced a weighting factor (1-$g(\mathbf{k})$) which is the availability of the states in the volume $\Delta\mathbf{k}$ centered in $\mathbf{k}$ (they should not be already occupied, because the Pauli principle must be satisfied).

Keep in mind that these scattering events are supposed to be LOCALIZED (depending only on $\mathbf{r}$ and t, not on $\mathbf{r}$' and t'). The **total balance** of the scattering events gives:

$$\left(\frac{\partial g}{\partial t}\right)_{coll} = -\int \frac{\Delta \mathbf{k}'}{8\pi^3} \left\{ W_{\mathbf{k}\mathbf{k}'} g(\mathbf{k}) \left[1 - g(\mathbf{k}')\right] - W_{\mathbf{k}'\mathbf{k}} g(\mathbf{k}')[1 - g(\mathbf{k})] \right\} \tag{18}$$

4 Relaxation time approximation

If we consider now that the scattering OUT events depend only locally on the distribution $g(\mathbf{k})$ around $\mathbf{k}$, and that the scattering IN events depend only on the *local equilibrium* distribution function $g^0(\mathbf{k})$ (local equilibrium *prior to* the collisions), we can dramatically simplify the IN and OUT collisions terms as follows:

$$\left(\frac{\partial g}{\partial t}\right)_{coll}^{OUT} = -\frac{g(\mathbf{k})}{\tau(\mathbf{k})} \tag{19}$$

$$\left(\frac{\partial g}{\partial t}\right)_{coll}^{IN} = \frac{g^0(\mathbf{k})}{\tau(\mathbf{k})} \tag{20}$$

where $\tau(\mathbf{k})$ is some *relaxation time*. The final result:

$$\left(\frac{\partial g}{\partial t}\right)_{coll} \approx -\frac{g(\mathbf{k}) - g^0(\mathbf{k})}{\tau(\mathbf{k})} \tag{21}$$

which gives the Boltzmann equation in this form:

$$\boxed{\frac{\partial}{\partial t} g + \frac{\partial}{\partial \mathbf{r}} g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial \mathbf{k}} g \cdot \frac{\mathbf{F}}{\hbar} = -\frac{\delta g}{\tau(\mathbf{k})}} \tag{22}$$

suggests that collisions tend to restore the equilibrium, balancing the effect of the drift terms.

5 Scattering from isotropic materials

In case of *isotropic materials* and *stationary state* and keeping the relaxation time approximation, it is convenient to focus on the *variation* of $g(\mathbf{k})$, by defining:

$$\delta g(\mathbf{k}) = g(\mathbf{k}) - g^0(\mathbf{k}) \tag{23}$$

In this case the equilibrium distribution function $g^0(\mathbf{k})$ is dependent *neither on* $\mathbf{r}$ *nor on* t, but only on $\mathbf{k}$. Therefore, since:

$$\frac{\partial \delta g}{\partial t} = \frac{\partial g}{\partial t} \tag{24}$$

$$\frac{\partial \delta g}{\partial \mathbf{r}} = \frac{\partial g}{\partial \mathbf{r}} \tag{25}$$

$$\frac{\partial \delta g}{\partial \mathbf{k}} = \frac{\partial (g - g^0)}{\partial \mathbf{k}} \tag{26}$$

we finally get that the Boltzmann equation reduces to:

$$\boxed{\frac{\partial}{\partial t}\delta g + \frac{\partial}{\partial \mathbf{r}}\delta g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial \mathbf{k}}(g^0 + \delta g) \cdot \frac{\mathbf{F}}{\hbar} = -\frac{\delta g}{\tau(\mathbf{k})}} \tag{27}$$

6 Applications and examples

Remaining within the assumption of SMALL FIELDS, we will discuss the following cases:

- Isotropic perturbations $\Longrightarrow$ we can use Eq. 27
  - D.C., static and uniform $\mathbf{E}$ (stationary state)
  - A.C., using Linear Response Theory
- Materials with $\nabla_{\mathbf{r}} T \neq 0$ and $\mu = \mu(\mathbf{r}) \Longrightarrow$ we must use Eq. 22
  - D.C., static and uniform $\mathbf{E}$ (stationary state)
  - $\nabla T = 0$ but $\mu = \mu(\mathbf{r})$; applied static and uniform $\mathbf{E}$ and $\mathbf{H}$

6.1 Isotropic perturbation: static and uniform E applied (stationary state)

In case of stationary state with static and uniform $\mathbf{E}$ applied, Eq. 27 further simplifies, since $\frac{\partial}{\partial t}\delta g{=}0$ (stationary state) and $\frac{\partial}{\partial \mathbf{r}}\delta g{=}0$ (deviations from the equilibrium cannot depend on the spatial point). Furthermore, $\frac{\partial}{\partial \mathbf{k}}\delta g \cdot \mathbf{E}$ must be neglected, being infinitesimal of second order in $\mathbf{E}$, so that the gradient in $\mathbf{k}$ contains only the contribution:

$$\frac{\partial}{\partial \mathbf{k}} g^0 = \nabla_{\mathbf{k}} \mathcal{E}(\mathbf{k}) \frac{\partial g^0}{\partial \mathcal{E}} = \hbar \mathbf{v}(\mathbf{k}) \frac{\partial g^0}{\partial \mathcal{E}} \tag{28}$$

Therefore we get:

$$\delta g = e\mathbf{E} \cdot \mathbf{v}(\mathbf{k}) \tau(\mathbf{k}) \frac{\partial g^0}{\partial \mathcal{E}} \tag{29}$$

The equation that relates current and electron velocity, which is $\mathbf{j} = -ne\mathbf{v}$ in the Drude model, becomes:

$$\mathbf{j} = -e \sum_{\mathbf{k}} \delta g(\mathbf{k}) \mathbf{v}(\mathbf{k}) = -e^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \left(\mathbf{E} \cdot \mathbf{v}(\mathbf{k})\right) \mathbf{v}(\mathbf{k}) \tau(\mathbf{k}) \frac{\partial g^0}{\partial \mathcal{E}} \tag{30}$$

We can see from the previous eq. the tensorial character of the conductivity:

$$j_i = \sum_j \sigma_{ij} E_j \tag{31}$$

with

$$\sigma_{ij} = e^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \tau(\mathbf{k}) v_i(\mathbf{k}) v_j(\mathbf{k}) \left[-\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})}\right] \tag{32}$$

Note:

- **Anisotropy:** in general $\mathbf{j}$ is NOT parallel to $\mathbf{E}$, at variance with free electron case. In case of *cubic* materials, however, the expression of σ is simplified: $\sigma_{ij} = \delta_{ij}\sigma$.
- **Filled bands are inert:** only deviations from full filling are important
- **Importance of Fermi surfaces:** Although the integral seems to be done over the whole BZ, the term $\left[\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})}\right]$ under the integral is non zero only around the Fermi surface (since it is basically $\propto \delta(\mathcal{E} - \mathcal{E}_F)$). The conductivity is determined by the conduction band in an interval of size $k_B T$ around $\mathcal{E}_F$.
- **Particular case: parabolic band and effective mass.** Considering that: (i) $-\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})} = \delta(\mathcal{E} - \mathcal{E}_F)$; (ii) averaging over all the electrons, $\langle v_i v_j \rangle = \langle v_i^2 \rangle \delta_{ij} = \frac{1}{3} v^2 \delta_{ij}$; (iii) accounting for a spin factor of 2 if the two spin channels equally contribute to the current; we have:

$$
\begin{aligned}
\sigma &= 2e^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \tau(\mathbf{k}) \frac{1}{3} v^2(\mathbf{k}) \delta(\mathcal{E} - \mathcal{E}_F) \\
&= \frac{e^2}{12\pi^3} \int_{\substack{Fermi \\ surf}} \tau(\mathbf{k}) v^2(\mathbf{k}) \frac{dS}{|\nabla_{\mathbf{k}} \mathcal{E}(\mathbf{k})|} \\
&= \frac{e^2}{12\pi^3} \int_{\substack{Fermi \\ surf}} \tau(\mathbf{k}) v(\mathbf{k}) \frac{1}{\hbar} dS
\end{aligned}
\tag{33}
$$

Furthermore, considering that in case of parabolic and isotropic band, the Fermi surface is a Fermi sphere with radius k_F and the integral of dS over such surface simply gives the surface area, we finally get:

$$
\sigma = \frac{e^2}{12\pi^3} \tau_F v_F \frac{1}{\hbar} 4\pi k_F^2 = \frac{n e^2 \tau_F}{m^*} \tag{34}
$$

which resembles the well known result for free electrons, BUT with $m \to m^*$ and $\tau \to \tau_F$. Attention: in the final result, the fact that ONLY the electrons around $\mathcal{E}_F$ do contribute to the current is hidden, but it is clear by following the derivation!

6.2 Isotropic perturbation: A.C. conductivity, using Linear Response Theory

A&M, Ch 13, p. 252

We start again from the linearized eq. 27 for δg, but since now $\mathbf{E}$ is time dependent, we must keep all the terms in $\frac{\partial}{\partial t}$. As in the previous case, $\frac{\partial}{\partial \mathbf{k}} \delta g \cdot \mathbf{E}$ must be neglected, being infinitesimal of second order in $\mathbf{E}$, so that we get:

$$\frac{\partial}{\partial t}\delta g + \frac{\partial}{\partial \mathbf{r}}\delta g \cdot \mathbf{v}(\mathbf{k}) - \frac{e\mathbf{E}}{\hbar} \cdot \frac{\partial g^0}{\partial \mathbf{k}} = -\frac{\delta g}{\tau(\mathbf{k})} \tag{35}$$

Consider $\mathbf{E}(\mathbf{r}, t) = \mathbf{E_0} e^{i(\mathbf{q}\cdot\mathbf{r} - wt)}$. In case of linear response regime, we have: $\delta g(\mathbf{r}, \mathbf{k}, t) = \Phi(\mathbf{k}) e^{i(\mathbf{q}\cdot\mathbf{r} - wt)}$. Substituting in eq. 35, we get:

$$-iw\Phi + \mathbf{v} \cdot i\mathbf{q}\Phi - \frac{e}{\hbar}\mathbf{E_0}\frac{\partial g^0}{\partial \mathbf{k}} = -\frac{\Phi}{\tau} \tag{36}$$

from which, using eq. 28, we obtain:

$$\Phi(\mathbf{k}) = \frac{e\tau \mathbf{E_0} \cdot \mathbf{v}}{1 - i\tau(w - \mathbf{q} \cdot \mathbf{v})}\frac{\partial g^0}{\partial \mathcal{E}} \tag{37}$$

Considering the expression for the current in terms of the electron velocity (always remind the analogous in the Drude model, $\mathbf{j} = -ne\mathbf{v}$):

$$\begin{aligned}
\mathbf{j} &= -e\int \frac{d\mathbf{k}}{(2\pi)^3}\delta g(\mathbf{k})\mathbf{v}(\mathbf{k}) \\
&= -e\int \frac{d\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})e^{i(\mathbf{q}\cdot\mathbf{r} - wt)}\mathbf{v}(\mathbf{k}) \\
&= -e\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{e\tau \mathbf{E} \cdot \mathbf{v}}{1 - i\tau(w - \mathbf{q} \cdot \mathbf{v})}\frac{\partial g^0}{\partial \mathcal{E}}\mathbf{v}(\mathbf{k})
\end{aligned} \tag{38}$$

and the expression for the current in terms of conductivity and electric field (eq. 31), we get:

$$\sigma_{ij} = e^2\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{\tau(\mathbf{k})v_i(\mathbf{k})v_j(\mathbf{k})}{1 - i\tau(\mathbf{k})\left(w - \mathbf{q} \cdot \mathbf{v}(\mathbf{k})\right)}\left[-\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})}\right] \tag{39}$$

which reduces to the static case previously discussed (eq. 32) when $\mathbf{q} \to 0$ (long wavelengths, field uniform in space) and $w \to 0$ (static limit).

6.3 Anisotropic material ($\nabla T \neq 0$, $\mu = \mu(\mathbf{r})$) static and uniform E (stationary state)

In this case also g^0 depends on $\mathbf{r}$, so that we must use eq. 22. We have:

$$\frac{\partial g}{\partial \mathbf{r}} = \frac{\partial g}{\partial \mu} \cdot \nabla\mu + \frac{\partial g}{\partial T} \cdot \nabla T \tag{40}$$

and since, being consistent with the linear expansion in the perturbative terms:

$$\begin{aligned}
\frac{\partial g}{\partial \mu} &\approx \frac{\partial g^0}{\partial \mu} = \frac{g^0}{k_B T}e^{\frac{\mathcal{E}-\mu}{k_B T}} \\
\frac{\partial g}{\partial T} &\approx \frac{\partial g^0}{\partial T} = \frac{\mathcal{E} - \mu}{(k_B T)^2}(g^0)^2 k_B e^{\frac{\mathcal{E}-\mu}{k_B T}}
\end{aligned} \tag{41}$$

Then, since $\frac{\partial g}{\partial \mathcal{E}} \approx \frac{\partial g^0}{\partial \mathcal{E}} = -\frac{(g^0)^2}{k_B T} e^{\frac{\mathcal{E}-\mu}{k_B T}}$, we can express the right-hand side terms of eq. 41 in terms of $\frac{\partial g^0}{\partial \mathcal{E}}$, obtaining:

$$\begin{aligned} \frac{\partial g}{\partial \mu} &\approx -\frac{\partial g^0}{\partial \mathcal{E}} \\ \frac{\partial g}{\partial T} &\approx -\frac{\mathcal{E}-\mu}{T} \frac{\partial g^0}{\partial \mathcal{E}} \end{aligned} \tag{42}$$

Similarly, also $\frac{\partial g}{\partial \mathbf{k}}$ can be expressed in terms of $\frac{\partial g^0}{\partial \mathcal{E}}$, obtaining:

$$\frac{\partial g}{\partial \mathbf{k}} = \frac{\partial g}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathbf{k}} \approx \frac{\partial g^0}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathbf{k}} = \frac{\partial g^0}{\partial \mathcal{E}}\ \hbar \mathbf{v}(\mathbf{k}). \tag{43}$$

Substituting in eq. 22 and considering the stationary regime $\left(\frac{\partial g}{\partial t} = 0\right)$, we get, a part from terms of higher order in $\nabla_{\mathbf{r}} T$ and $\mathbf{E}$:

$$\tau(\mathcal{E}(\mathbf{k}))\ \mathbf{v}(\mathbf{k}) \cdot \left(\nabla_{\mathbf{r}}\mu + \frac{\mathcal{E}-\mu}{T}\ \nabla_{\mathbf{r}} T + e\mathbf{E}\right)\left(-\frac{\partial g^0}{\partial \mathcal{E}}\right) = g^0(\mathbf{k}) - g(\mathbf{k}) \tag{44}$$

which is eq. (13.43) of A&M book (Ch. 13.2).

6.4 Anisotropic material ($\nabla T = 0$ but $\mu = \mu(\mathbf{r})$) static and uniform E and H (stationary state)

For the sake of simplicity, we consider now $\nabla T = 0$. We can start from the last eq. 44. The external force term, $e\mathbf{E}$, must be substituted with $e\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}$. It is interesting to notice that $\mathbf{v} \cdot \mathbf{v} \times \mathbf{H} = 0$, so that the external applied magnetic field has NO EFFECT! Furthermore, if we consider the case of a parabolic and isotropic band, so that $\mathcal{E}(\mathbf{k}) = \frac{\hbar^2 k^2}{2m^*}$, we have that $\mathbf{v} = \frac{\hbar \mathbf{k}}{m^*}$ (note that only in this approximation $\mathbf{v}$ is parallel to $\mathbf{k}$!), and eq. 44 becomes:

$$\underbrace{\tau(\mathcal{E}(\mathbf{k}))\ \left(-\frac{\partial g^0}{\partial \mathcal{E}}\right) (\nabla_{\mathbf{r}}\mu + e\mathbf{E})\ \frac{\hbar}{m^*}} \cdot \mathbf{k} = g^0(\mathbf{k}) - g(\mathbf{k}). \tag{45}$$

The part $\underbrace{\quad}$ in the left-hand side term is dependent only on $\mathcal{E}$, so that we can rewrite the last equation in a very simplified form:

$$\mathbf{a}(\mathcal{E}) \cdot \mathbf{k} = g^0(\mathbf{k}) - g(\mathbf{k}). \tag{46}$$

which is eq. (16.26) of A&M book.