1 APPENDIX B

NOETHER THEOREM FOR POINT PARTICLE-DYNAMICS

1.1 Introduction

The Noether theorem is a precise derivation of the fact that once a Lagrangian system has a *symmetry* there is automatically a *conserved quantity*. The Noether theorem give also an exact expression of that conserved quantity.

Emmy Noether gave a derivation of this theorem only after Einstein had proposed the theory of general relativity (GR). It was Hilbert, after having contributed to the formulation of the equations of motion of general relativity, who suggested to E.Noether to understand which were the conserved quantities associated to the many symmetries of GR. So Noether theorem is something that was developed relatively recently, even if its simplified version was somehow known in the form we presented in the notes of the course.

In this appendix we will give a derivation taken from the review paper: E.L.Hill, Reviews of Modern Physics, vol.23, no.3 (1951) 253.

1.2 Symmetries

Let us suppose we work with a mechanical system whose configuration space has coordinates (q) (we will drop indices for the moment) and equations of motion of the form:

$$\ddot{q} = G(q, \dot{q}, t). \tag{1}$$

We define as *symmetry* a transformation on q:

$$q \longrightarrow q', \tag{2}$$

which keeps the *form* of the equation of motion invariant,i.e.:

$$\ddot{q}' = G(q', \dot{q}', t). \tag{3}$$

If the equations of motion can be derived from a variational principles, then we must have a Lagrangian $\mathcal{L}$, and in the new coordinates one expects that the Lagrangian changes in form into a new one: $\mathcal{L}'$. On the other hand the action must maintain the same values in the two coordinates because it is just a real number, this implies that

$$\int_{t_1}^{t_2} \mathcal{L}'(q', \dot{q}')dt = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q})dt, \tag{4}$$

This is basically the definition of $\mathcal{L}'$.

If the equations of motion, expressed in the new coordinates, must maintain the same form (3), then the two Lagrangian must differ by a total derivative at most:

$$\mathcal{L}'(q', \dot{q}') = \mathcal{L}(q', \dot{q}') + \frac{d\Omega(q')}{dt}. \tag{5}$$

The proof of this is easy and it goes as follows. We know that one can get equation of motion by minimizing the action on the set of trajectories which have fixed initial and final points:

$$\delta S \equiv \delta \int_{(q_1,t_1)}^{(q_2,t_2)} \mathcal{L} dt = 0. \tag{6}$$

where the δ indicates the variational deformation of the paths and not the infinitesimal symmetry transformation we will introduce in a moment. Using the expression in (5), we obtain for the variation above:

$$\delta \int_{(q'_1,t_1)}^{(q'_2,t_2)} \mathcal{L}'(q', \dot{q}') dt = \delta \int_{(q'_1,t_1)}^{(q'_2,t_2)} \mathcal{L}(q', \dot{q}') dt + \delta \int_{(q'_1,t_1)}^{(q'_2,t_2)} \frac{d\Omega(q')}{dt} dt \tag{7}$$

If we use, on the LHS of the equation above, the relation (4) we get:

$$\delta \int_{(q_1,t_1)}^{(q_2,t_2)} \mathcal{L}(q, \dot{q}) dt = \delta \int_{(q'_1,t_1)}^{(q'_2,t_2)} \mathcal{L}(q', \dot{q}') dt + \delta[\Omega(q'_2) - \Omega(q'_1)]. \tag{8}$$

As the end points are fixed under the variational deformation, the last term on the RHS of (8) is zero:

$$\delta[\Omega(q'_2) - \Omega(q'_1)] = 0 \tag{9}$$

and so eq.(8) becomes :

$$\delta \int_{(q_1,t_1)}^{(q_2,t_2)} \mathcal{L}(q, \dot{q}) dt = \delta \int_{(q'_1,t_1)}^{(q'_2,t_2)} \mathcal{L}(q', \dot{q}') dt. \tag{10}$$

From this relation we get that the equations of motion maintain the same form in the two different coordinates.

The relation (4) plus the (5) are the two (sufficient) conditions we have to impose in order to have a symmetry and they will be the starting point to derive the Noether theorem.

1.3 Derivation of the Noether Theorem

Before embarking in that, we should clarify some issues related to the infinitesimal variations which will be used in the proof. The first variation we will consider is the following:

$$q'(t') = q(t) + \delta q \quad , \quad t' = t + \delta t. \tag{11}$$

Note that here we have supposed that not only q but even the time t changes under our symmetry trasformation. In case we want the variation at the same instant of time, we shall indicate the variation with the symbol $\bar{\delta}$:

$$q'(t) = q(t) + \bar{\delta}q. \tag{12}$$

Actually, remembering the expressions above of δq and $\bar{\delta}q$, we can easily derive a relation among the two:

$$\begin{aligned} \delta q(t) &\equiv q'(t') - q(t) = \\ &= q'(t) + \dot{q}'(t)\delta t - q(t) \\ &= \bar{\delta}q + \dot{q}(t)\delta t \end{aligned} \tag{13}$$

It will also be usefull to compare the variations δ and $\bar{\delta}$ of the derivatives of q.It is easy to prove that the derivative $\frac{d}{dt}$ commutes with the $\bar{\delta}q$, in fact :

$$\begin{aligned} \frac{d}{dt}\bar{\delta}q(t) &= \frac{d}{dt}[q'(t) - q(t)] = \\ &= \frac{d}{dt}q'(t) - \frac{d}{dt}q(t) = \bar{\delta}\dot{q} \end{aligned} \tag{14}$$

This does not happen with the δq, in fact let us first calculate $\delta\dot{q}$:

$$\begin{aligned} \delta\dot{q} &\equiv \dot{q}'(t') - \dot{q}(t) \\ &= \dot{q}'(t) + \ddot{q}\delta t - \dot{q}(t) \\ &= \bar{\delta}\dot{q}(t) + \ddot{q}\delta t \end{aligned} \tag{15}$$

If we now calculate $\frac{d}{dt}\delta q$ starting from eq. (13), we get

$$\frac{d}{dt}\delta q = \bar{\delta}\dot{q} + \ddot{q}\delta t + \dot{q}\frac{d\delta t}{dt} \tag{16}$$

We immediately notice that the RHS of (16) and (15) are not the same.

Let us now turn to the proof of the Noether theorem. If, besides q, we have a change also in t the two eqs.(4) and (5) can be combined in the following equation, (where for simplicity we have omitted the extremes of integration):

$$\int \mathcal{L}(q', \dot{q}', t')dt' - \int \mathcal{L}(q, \dot{q}, t)dt = -\int \frac{d\Omega}{dt}dt. \tag{17}$$

Note that above, in making use of eq.(4), we have changed also the time in the integration on the LHS of (4) because now our transformation is the one given by eq.(11). Moreover in the function Ω we have used, in (17), the variable q and not the q'. This is due to the fact that Ω already contains an infinitesimal parameter, as it is clear from

(5), so the difference in writing Ω with either q or q' is second order in the infinitesimal parameter and can be neglected.

Let us now expand, in the first Lagrangian of eq.(17), the variables $(q', \dot{q}', t')$ in term of $(q, \dot{q}, t)$ and their variations, we get that eq.(17) becomes:

$$\int [\mathcal{L} + \frac{\partial \mathcal{L}}{\partial t}\delta t + \frac{\partial \mathcal{L}}{\partial q}\delta q \quad + \quad \frac{\partial \mathcal{L}}{\partial \dot{q}}\ \delta \dot{q}\ ]\ dt' + \\ - \int \mathcal{L}(q, \dot{q}, t)\ dt = -\int \frac{d\Omega}{dt}\ dt. \tag{18}$$

On the LHS of eq.(18) let us rewrite the integration volume dt' in term of dt and δt from eq.(11):

$$dt' = dt[1 + \frac{\partial(\delta t)}{\partial t}], \tag{19}$$

and next let us rewrite, on the LHS of eq.(18), the δq and $\delta \dot{q}$ in terms of $\bar{\delta} q$ and $\bar{\delta} \dot{q}$ using the relations (13) and (15). What we get is:

$$\int [\mathcal{L}\frac{\partial \delta t}{\partial t} + \frac{\partial \mathcal{L}}{\partial t}\ \delta t + \frac{\partial \mathcal{L}}{\partial q}\ \bar{\delta} q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ \bar{\delta} \dot{q} + \frac{\partial \mathcal{L}}{\partial q}\ \dot{q}\ \delta t + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ \ddot{q}\ \delta t\ ]\ dt = -\int \frac{d\Omega}{dt}\ dt. \tag{20}$$

Before proceeding, let us remember that for a function (like the Lagrangian) of $(q, \dot{q}, t)$, the partial derivative $\frac{\partial}{\partial t}$ is related to the total derivative $\frac{d}{dt}$ as follows:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \ddot{q}\frac{\partial}{\partial \dot{q}} \tag{21}$$

If we now make use of the relation (21) in eq.(20) we get:

$$\int [\mathcal{L}\frac{\partial \delta t}{\partial t} + (\frac{d}{dt} - \dot{q}\frac{\partial}{\partial q} \quad - \quad \ddot{q}\frac{\partial}{\partial \dot{q}})\mathcal{L}\ \delta t + \frac{\partial \mathcal{L}}{\partial q}\ \bar{\delta} q + \\ + \quad \frac{\partial \mathcal{L}}{\partial \dot{q}}\ \bar{\delta} \dot{q} + \frac{\partial \mathcal{L}}{\partial q}\ \dot{q}\ \delta t + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ \ddot{q}\ \delta t\ ]\ dt = -\int \frac{d\Omega}{dt}\ dt. \tag{22}$$

The above equation can be simplified into the following one:

$$\int \left[\frac{d}{dt}\left(\mathcal{L}\delta t\right) + \frac{\partial \mathcal{L}}{\partial q}\ \bar{\delta} q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ \bar{\delta} \dot{q}\right] dt = -\int \frac{d\Omega}{dt} dt. \tag{23}$$

Rewriting the last term on the LHS of the equation above as:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}}\bar{\delta} \dot{q} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\bar{\delta} q\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\bar{\delta} q, \tag{24}$$

and inserting it into eq.(23), we get:

$$\int \left[\frac{d}{dt}\left(\mathcal{L}\delta t\right) + \frac{\partial \mathcal{L}}{\partial q}\bar{\delta} q - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\ \bar{\delta} q + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\bar{\delta} q\right)\right] dt = -\int \frac{d\Omega}{dt} dt \tag{25}$$

Indicating with the symbol $\Delta\mathcal{L}$ the Lagrangian eqs. of motion, i.e.:

$$\Delta\mathcal{L} \equiv \frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}}, \tag{26}$$

we can rewrite the eq.(25) as follows

$$\int\left[\frac{d}{dt}(\mathcal{L}\delta t) + (\Delta\mathcal{L})\bar{\delta}q + \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\bar{\delta}q\right)\right]dt = -\int\frac{d\Omega}{dt}dt \tag{27}$$

or equivalently as:

$$\int\left[\frac{d}{dt}\left(\mathcal{L}\delta t + \frac{\partial\mathcal{L}}{\partial\dot{q}}\bar{\delta}q\right) + \frac{d\Omega}{dt}\right]\, dt = -\int(\Delta\mathcal{L})\bar{\delta}q\; dt, \tag{28}$$

In the equation above we can replace $\bar{\delta}q$ with its expression in term of δq as given in eq.(13): $\delta\bar{q} = \delta q - \dot{q}\delta t$, and so we can re-write eq.(28) as:

$$\int\frac{d}{dt}\left[\left(\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\dot{q}}\dot{q}\right)\delta t + \frac{\partial\mathcal{L}}{\partial\dot{q}}\;\delta q + \Omega\right]dt = -\int\left[(\Delta\mathcal{L})\,\bar{\delta}q\right]dt. \tag{29}$$

From the whole derivation presented so far it is clear that the above equation (29) holds for any interval of time we use in the integral, that implies that the integrands on the RHS and LHS of (29) must be equal, i.e.:

$$\frac{d}{dt}Q = -(\Delta\mathcal{L})\bar{\delta}q, \tag{30}$$

where :

$$Q \equiv \left(\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\dot{q}}\dot{q}\right)\delta t + \frac{\partial\mathcal{L}}{\partial\dot{q}}\delta q + \Omega. \tag{31}$$

If the classical eqs. of motion hold, i.e. $\Delta\mathcal{L} = 0$, then eq.(30) becomes:

$$\frac{dQ}{dt} = 0. \tag{32}$$

This equations tells us that, once we have a symmetry in a Lagrangian system, there is automatically an associated charge (31) which is conserved (i.e. eq.(32)).

This is the content of the Noether theorem which, besides proving the statement above, gives us also an explicit expression (31) of the conserved charge. Note that the Q in (31) is actually a function of $(q,\dot{q},t)$ multiplied by the infinitesimal parameters $(\delta q, \delta t)$ and what we properly call charge is actually the function stripped of the infinitesimal parameters.

If there are more than one degree of freedom in our system, the charge (31) would have the expression:

$$Q \equiv \left(\mathcal{L} - \sum_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{q}_i\right)\delta t + \sum\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta q_i + \Omega. \tag{33}$$

1.4 Explicit Examples

Let us now study a system of point particles interacting one with the other and, for simplicity, let us suppose that the interaction depends only on the relative distance among the particles. So the Lagrangian would have the form:

$$\mathcal{L} = \sum_i \frac{1}{2}\left(\dot{X}_i^2 + \dot{Y}_i^2 + \dot{Z}_i^2\right) - V\left(\mid \vec{R}_{ij} \mid\right), \tag{34}$$

where the subindex "i" labels the various particles and

$$\left(\mid \vec{R}_{ij} \mid\right)^2 \equiv (X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2 \tag{35}$$

SPACE-TRANSLATION

Let us perform on the Lagrangian the following infinitesimal transformation which is a translation in the X coordinates of the whole system:

$$X_i' = X_i - \zeta \ , \ Y_i' = Y_i \ , \ Z_i' = Z_i \ , \ t' = t. \tag{36}$$

ζ is an infinitesimal quantity which does not depend on the index "i" and on X, Y, Z, t, and as a consequence:

$$\dot{X}_i' = \dot{X}_i \ , \ \dot{Y}_i' = \dot{Y}_i \ , \ \dot{Z}_i' = \dot{Z}_i. \tag{37}$$

It is now easy to check that the Lagrangian (34) is left invariant by the transformations (36),(37), so this is a symmetry. Moreover it is also easy to prove that the Ω-function of (5) is zero.

The conserved charge (33), as $\delta t = 0$, turns out to be:

$$Q = \sum \frac{\partial \mathcal{L}}{\partial \dot{X}_i}\delta X_i = \sum \frac{\partial \mathcal{L}}{\partial \dot{X}_i}\zeta = \zeta \sum P_i^{(x)}, \tag{38}$$

where we have indicated with $P_i^{(x)}$ the momenta along X of the particle "i".

So the conserved charge (38) is, once stripped of the infinitesimal parameter ζ, the total momentum along X of the system. If we had performed a translation along Y or Z we would have analogously had the conservation of the total momenta along Y or Z. Note that the momenta of the single particle is not conserved due to the interaction among the particles.

TIME TRANSLATION

The Lagrangian (34) is invariant also under time translation which is formally the following transformation:

$$\delta t = \epsilon \ , \ \delta X_i = 0 \ , \ \delta Y_i = 0 \ , \ \delta Z_i = 0, \tag{39}$$

where ϵ is an infinitesimal parameter independent of "(i)" and of (X, Y, Z, t). Because of this we have that:

$$\delta \dot{X}_i = 0 \ , \ \delta \dot{Y}_i = 0 \ , \ \delta \dot{Z}_i = 0. \tag{40}$$

It is easy to check that the Ω-function (5) is zero for this transformation. Note that this invariance would not be there only if the potential V in (34) depended explicitly on t.

Inserting now eq.(39) and (40) in (31) we get:

$$Q = \left[\mathcal{L} - \sum \left(\frac{\partial \mathcal{L}}{\partial \dot{X}_i}\dot{X}_i + \frac{\partial \mathcal{L}}{\partial \dot{Y}_i}\dot{Y}_i + \frac{\partial \mathcal{L}}{\partial \dot{Z}_i}\dot{Z}_i\right)\right]\epsilon = -H\epsilon, \tag{41}$$

where H is the Hamiltonian of the system. So, as we can identify the Hamiltonian with the energy, we can say that the conserved quantity associate to the symmetry of time-translation is the the <u>total energy</u> of the system.

<u>*SPACE-ROTATION*</u>

The Lagrangian of our system (34) is built out of only the modulus of vectors, that are the velocity of the particles and the relative distances among particles. As a consequence the whole Lagrangian is invariant under space rotation because the lengths of vectors are kept invariant by space-rotation.

Let us now find out out which is the conserved quantity associated to this symmetry. To simplify things let us just analyze the rotation around the Z axis whose infinitesimal transformations are:

$$X_i' = X_i + \alpha Y_i \ , \ Y_i' = Y_i - \alpha X_i \ , \ Z_i' = Z_i \ , \ t' = t, \tag{42}$$

where α is the infinitesimal angle of rotation and we suppose it does not depend on "i" and on (X, Y, Z, t), and as a consequence the transformations of the velocities are:

$$\dot{X}_i' = \dot{X}_i + \alpha \dot{Y}_i \ , \ \dot{Y}_i' = \dot{Y}_i - \alpha \dot{X}_i \ , \ \dot{Z}_i' = \dot{Z}_i \tag{43}$$

Like for the previous symmetries it is easy to check that the Ω-function of (5) is zero and so inserting eq.(42) and (43) into (33), we get

$$Q = \sum \frac{\partial \mathcal{L}}{\partial \dot{X}_i} Y_i \alpha - \frac{\partial \mathcal{L}}{\partial \dot{Y}_i} X_i \alpha = \alpha \sum P_i^{(x)} Y_i - P_i^{(y)} X_i = \alpha \sum L_i^{(z)} \tag{44}$$

This is the total angular momentum along the Z axis. An analog expression along X or Y would be obtained if we had performed a rotation along X or along Y. So we can say that the invariance under rotation leads to the conservation of the <u>angular momentum</u> .

The reader may wonder why we called the conserved quantities as "charges". The habit comes from field theory where it is possible to prove that the even the conservation of the "charges" of the particles comes from a symmetry.