An introduction to quantum spin liquids: definitions and examples

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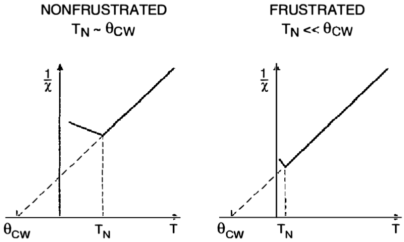
Current Trends in Frustrated Magnetism
Jawaharlal Nehru University (JNU), 8 February 2015

Searching for non-magnetic ground states

- In a spin model, magnetic order is expected at (mean field):

$$k_B T_N \propto z S(S+1)|J|$$

z is the coordination number, S is the spin and J is the super-exchange coupling



$$\chi = \frac{C}{T - \theta_{cw}} \qquad T \gg T_N$$

θ_{CW} is the Curie-Weiss temperature

$$f = \frac{|\theta_{cw}|}{T_N}$$

- Can quantum fluctuations prevent magnetic order down to $T = 0$?

$\Longrightarrow$ Look for low spin S, low coordination z, competing interactions:

Pomeranchuk, Zh. Eksp. Teor. Fiz. **11**, 226 (**1941**)

Ultimate frustration?

Looking for a magnetically disordered ground state

- Many theoretical suggestions since P.W. Anderson (1973)

  Anderson, Mater. Res. Bull. **8**, 153 (1973)

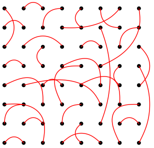
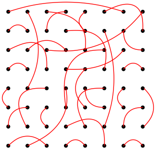
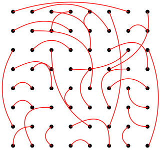
  Fazekas and Anderson, Phil. Mag. **30**, 423 (1974)

"Resonating valence-bond" (RVB) states

Idea: the best state for two spin-1/2 spins is a valence bond (a spin singlet):

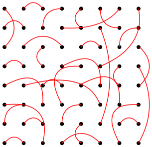
$$|VB\rangle_{\mathbf{R},\mathbf{R}'} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_{\mathbf{R}}|\downarrow\rangle_{\mathbf{R}'} - |\downarrow\rangle_{\mathbf{R}}|\uparrow\rangle_{\mathbf{R}'}\right)$$

Every spin of the lattice is coupled to a partner
Then, take a superposition of different valence bond configurations

$\Psi =$  $+$  $+$  $+ \ldots$

Valence-bond states: liquids and solids

Long-range RVB



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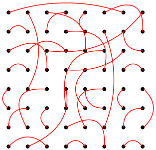
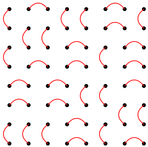


$+ \dots$

Short-range RVB



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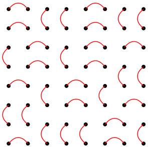
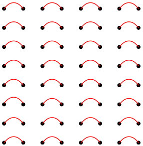


$+ \dots$

Valence-bond solid



breaks translational/rotational symmetries

General properties of valence-bond states

RVB states are typical examples of spin liquids

- The formation of a valence bond implies a gap to excite those two spins

- Long-range valence bonds are more weakly bound: a gapless spectrum is possible

  The projected Fermi sea can be seen as a long-range valence bond state:

  $|\Psi\rangle = \mathcal{P}_G \prod_{k<|k_F|} c^\dagger_{k,\uparrow} c^\dagger_{k,\downarrow} |0\rangle$

- It is now clear that the number of distinct quantum spin liquids is also huge
  hundreds of different quantum spin liquids have been classified
  (all with the same symmetry $\Longrightarrow$ topological order)

  Wen, Phys. Rev. B **65**, 165113 (2002)

- It is usually believed that such states may be described by gauge theories
  (at least at low energies/temperatures)

  Baskaran and Anderson, Phys. Rev. B **37**, 580 (1988)

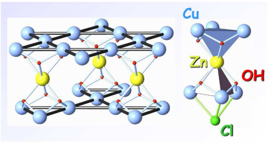
  $\Longrightarrow$ Gauge excitations should be visible in the spectrum!

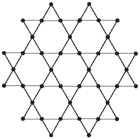
Candidate materials for $S = 1/2$ spin liquids

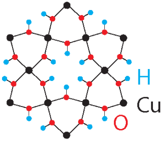
- Many experimental efforts to synthetize new materials

Two-dimensional Kagome lattice: Herbertsmithite and Volborthite

$ZnCu_3(OH)_6Cl_2$ and $Cu_3V_2O_7(OH)_2\ 2H_2O$

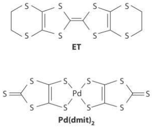


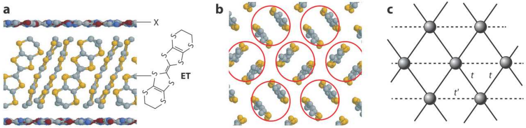




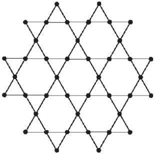
Two-dimensional anisotropic lattice: organic materials

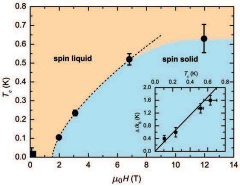
κ-$(BEDT\text{-}TTF)_2Cu_2(CN)_3$ and $EtMe_3Sb[Pd(dmit)_2]_2$

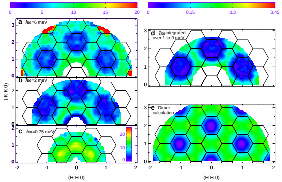




Candidate materials for $S = 1/2$ spin liquids



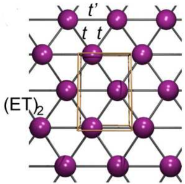


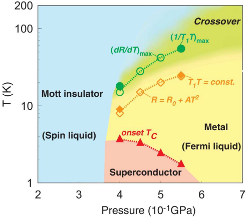


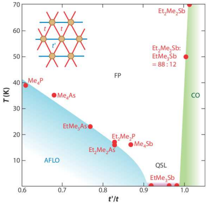
Jeong *et al.*, Phys. Rev. Lett. **107**, 237201 (2011)

de Vries *et al.*, Phys. Rev. Lett. **103**, 237201 (2009)

Han *et al.*, Nature **492**, 407 (2012)







Kanoda and Kato, Annu. Rev. Condens. Matter Phys. **2**, 167 (2011)

Shimizu *et al.*, Phys. Rev. Lett. **91**, 107001 (2003)

Candidate materials for $S = 1/2$ spin liquids

Material	Lattice	$\lvert\theta_{cw}\rvert$	f
κ-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$	$\approx$ triangular	375K	$> 10^3$
EtMe$_3$Sb[Pd(dmit)$_2$]$_2$	$\approx$ triangular	350K	$> 10^3$
$ZnCu_3(OH)_6Cl_2$	kagome	240K	$> 10^3$
$Cu_3V_2O_7(OH)_2 \cdot 2H_2O$	$\approx$ kagome	120K	≈ 100
$BaCu_3V_2O_8(OH)_2$	$\approx$ kagome	80K	$> 10^2$
Cs_2CuCl_4	quasi one-dimensional	4K	≈ 10

Microscopic Heisenberg models

Here, I will discuss **spin models** (frozen charge degrees of freedom)

- Quantum spins on the lattice
- Zero temperature, i.e., ground-state properties
- Mainly with SU(2) spin symmetry:

$$\mathcal{H} = \sum_{i,j} J_{i,j} \mathbf{S}_i \cdot \mathbf{S}_j$$

- ...but in some case also with a lower spin symmetry $U(1) \times Z_2$:

$$\mathcal{H} = \sum_{i,j} J^z_{i,j} S^z_i S^z_j + \sum_{i,j} J^{xy}_{i,j} \left(S^x_i S^x_j + S^y_i S^y_j \right)$$

- I will not discuss the effect of an external magnetic field (magnetization plateaux)

Absence of magnetic order in one dimension

In D=1 many exactly solvable models (e.g., Heisenberg and Haldane-Shastry)

Bethe, Z. Phys. **71**, 205 (1931)

Haldane, Phys. Rev. Lett. **60**, 635 (1988); Shastry, Phys. Rev. Lett. **60**, 639 (1988)

Simple example: the one-dimensional XY model:

$$\mathcal{H} = J \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) = \frac{J}{2} \sum_i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+)$$

- Representing spin operators via hard-core bosons

$$S_i^+ = b_i^\dagger \qquad S_i^- = b_i \qquad S_i^z = b_i^\dagger b_i - \frac{1}{2}$$

- Perform a Jordan-Wigner transformation

Jordan and Wigner, Z. Phys. **47**, 631 (1928)

$$b_j = c_j e^{i\pi \sum_{n<j} c_n^\dagger c_n} \quad \Longleftarrow \text{String}$$

c_i are (spinless) fermionic operators

$$\mathcal{H} = \frac{J}{2} \sum_i (c_i^\dagger c_{i+1} + h.c.)$$

Free fermions with gapless excitations

Ground state and excitations

$$\mathcal{H} = \frac{J}{2} \sum_i (c_i^\dagger c_{i+1} + h.c.)$$

Boundary conditions depend upon the number N of fermions (or bosons):
N odd $\Longrightarrow$ periodic boundary conditions
N even $\Longrightarrow$ anti-periodic boundary conditions

- Ground state (always unique because of the boundary conditions)

$$|\Psi_0\rangle = \prod_{|k|>k_F} c_k^\dagger |0\rangle$$

- Single-particle excitations

$$|\Psi_k\rangle = c_k |\Psi_0\rangle \qquad |k| > k_F$$

  does not live in the correct (bosonic) Hilbert space:
  One must also change boundary conditions: non-local operator
  $\Longrightarrow S_k^+$ or S_k^- do not create elementary excitations

- Particle-hole excitations

$$|\Psi_{k,q}\rangle = c_{k+q}^\dagger c_k |\Psi_0\rangle \qquad |k| > k_F \text{ and } |k+q| < k_F$$

  They are terribly complicated in terms of bosons (because of the string)!

Elementary excitations: the spinons

- In $D = 1$ systems, elementary excitations are spinons carrying $S = 1/2$

  Faddeev and Takhtajan, Phys. Lett. **85A**, 375 (1981)

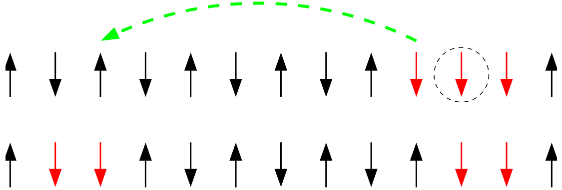
$$\mathcal{H} = J^z \sum_i S_i^z S_{i+1}^z + J^{xy} \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)$$

- A spinon is a neutral spin-1/2 excitation, "one-half" of a $S = 1$ spin flip.
- Spinons can only be created by pairs in finite systems

  They can propagate at large distances, as two elementary particles

FRACTIONALIZATION

Simple picture for $J^z \gg J^{xy}$ ($J^{xy} = 0$ corresponds to the Ising model)



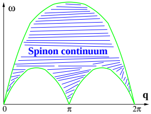
Elementary excitations: the spinons

- In $D = 1$ systems, elementary excitations are spinons carrying $S = 1/2$

Faddeev and Takhtajan, Phys. Lett. **85A**, 375 (1981)

$$S(q,\omega) = \int dt \langle \Psi_0 | S^z_{-q}(t) S^z_q(0) | \Psi_0 \rangle e^{i\omega t} = \sum_{n \neq 0} |\langle \Psi_n | S^z_q | \Psi_0 \rangle|^2 \delta(\omega - \Delta\omega_{n0})$$

$S(q,\omega)$ has only the incoherent part
No delta function
Singularity at the bottom of the spectrum



$S(q,\omega)$ can be computed exactly also in the Haldane-Shastry model:

$$\mathcal{H} = J \sum_{m<n} [d(m-n)]^2 \mathbf{S}_m \cdot \mathbf{S}_n \qquad d(n) = \frac{N}{\pi} \sin(\frac{\pi n}{N})$$

- The exact ground state is $|\Psi_0\rangle = \mathcal{P}_G \prod_{k<|k_F|} c^\dagger_{k,\uparrow} c^\dagger_{k,\downarrow} |0\rangle$

Haldane, Phys. Rev. Lett. **60**, 635 (1988); Shastry, Phys. Rev. Lett. **60**, 639 (1988)

- The $S = 1$ state $S^\alpha_n |\Psi_0\rangle$ is completely expressible in terms of two spinons

Haldane and Zirnbauer, Phys. Rev. Lett. **71**, 4055 (1993)

Dimerization in one dimension

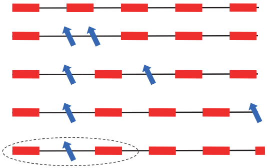
Majumdar-Ghosh point for the frustrated $J_1 - J_2$ Heisenberg model:

$$\mathcal{H} = J \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{J}{2} \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+2}$$

- The exact ground state is known (two-fold degenerate): perfect dimerization

$$\mathcal{H} = \frac{3J}{4} \sum_i \mathcal{P}_{3/2}^{(i-1,i,i+1)} - \frac{3J}{8} N \qquad \mathcal{P}_{3/2}^{(i-1,i,i+1)} = \frac{1}{3} \left[(\mathbf{S}_{i-1} + \mathbf{S}_i + \mathbf{S}_{i+1})^2 - \frac{3}{4} \right]$$

Majumdar and Ghosh, J. Math. Phys. **10**, 1388 (1969)



The "initial" $S = 1$ excitation can decay into two spatially separated spin-1/2 excitations

Finite-energy state with an isolated spinon (the other is far apart)

From one to two (and three) spatial dimensions

- In $D = 1$ there is:
  No magnetic order, given the Mermin-Wagner theorem
  (not possible to break a continuous symmetry in $D = 1$, even at $T = 0$)
  Pitaevskii and Stringari, J. Low Temp. Phys. **85**, 377 (1991)

  Fractionalization, e.g., $S = 1$ excitations decay into $S = 1/2$ particles

- In $D = 2$ it is possible to break a continuous symmetry at $T = 0$:

  Is it still possible to have a magnetically disordered ground state?

  Is it still possible to have fractionalization?

- In $D = 3$ it is possible to break a continuous symmetry even at finite temperature:

  Is it still possible to have a magnetically disordered ground state?

  Phase transition between two different paramagnetic phases at low temperature?

The semi-classical approach: large-S

$$\mathcal{H} = \sum_{i,j} J_{i,j} \mathbf{S}_i \cdot \mathbf{S}_j$$

- Suppose that the classical limit ($S \to \infty$) is described by

$$\mathbf{S}_i = \frac{1}{\sqrt{N}} \left(\mathbf{S}_{k_0} e^{ik_0 r_i} + h.c. \right) = \{\cos(k_0 r_i), \sin(k_0 r_i), 0\}$$

- In order to include the quantum fluctuations, perform a $1/S$ expansion
  - Let us denote by $\theta_j = k_0 r_j$
  - Make a rotation around the z axis

$$\begin{cases} \tilde{S}_j^x = \cos\theta_j S_j^x + \sin\theta_j S_j^y \\ \tilde{S}_j^y = -\sin\theta_j S_j^x + \cos\theta_j S_j^y \\ \tilde{S}_j^z = S_j^z \end{cases}$$

  - Perform the Holstein-Primakoff transformation:

$$\begin{cases} \tilde{S}_j^x = S - a_j^\dagger a_j \\ \tilde{S}_j^y \simeq \sqrt{\frac{S}{2}} \left(a_j^\dagger + a_j \right) \\ \tilde{S}_j^z \simeq i\sqrt{\frac{S}{2}} \left(a_j^\dagger - a_j \right) \end{cases}$$

The semi-classical approach: large-S

At the leading order in $1/S$, we obtain:

$$\mathcal{H}_{\mathrm{sw}} = \mathrm{E}_{\mathrm{cl}} + \frac{S}{2} \sum_k \left\{ A_k a_k^\dagger a_k + \frac{B_k}{2} \left(a_k^\dagger a_{-k}^\dagger + a_{-k} a_k \right) \right\}$$

Where:

$$\mathrm{E}_{\mathrm{cl}} = \frac{1}{2} N S^2 J_{k_0}$$

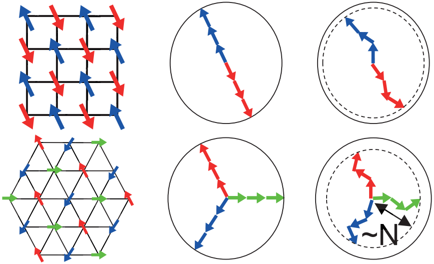
$$\begin{cases} A_k = J_k + \frac{1}{2}(J_{k+k_0} + J_{k-k_0}) - 2J_{k_0} \\ B_k = \frac{1}{2}(J_{k+k_0} + J_{k-k_0}) - J_k \end{cases}$$

By performing a Bogoliubov transformation:

$$\boxed{\mathcal{H}_{sw} = \mathrm{E}_{\mathrm{cl}} + \sum_k \omega_k (\alpha_k^\dagger \alpha_k + \tfrac{1}{2})}$$

- Leading-order corrections to the magnetization $\langle \tilde{S}_j^x \rangle = S - \langle a_j^\dagger a_j \rangle$
- Excitations are called magnons (analog of phonons for lattice waves)
- Presence of gapless excitations for broken SU(2) systems (Goldstone mode)

The classical ground state is "dressed" by quantum fluctuations



- The lattice breaks up into sublattices
- Each sublattice keeps an extensive magnetization

$$S(q) = \frac{1}{N} \langle \Psi_0 | \left| \sum_j \mathbf{S}_j e^{iqr_j} \right|^2 | \Psi_0 \rangle = \frac{1}{N} \sum_{j,k} \langle \Psi_0 | \mathbf{S}_j \cdot \mathbf{S}_k | \Psi_0 \rangle e^{iq(r_j - r_k)}$$

$$S(q) = \begin{cases} O(1) & \text{for all q's} \quad \rightarrow \text{short-range correlations} \\ S(q_0) \propto N & \text{for} q = q_0 \quad \rightarrow \text{long-range order} \end{cases}$$

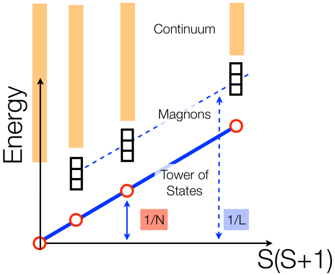
- Spontaneous symmetry breaking is only possible in the thermodynamic limit
  Spontaneously broken SU(2) symmetry $\Longrightarrow$ Gapless spin waves

- How can we detect it on finite lattices (e.g., by exact diagonalizations)?
  $\Longrightarrow$ Tower of states

Anderson, Phys. Rev. **86**, 694 (1952)

Bernu, Lhuillier, and Pierre, Phys. Rev. Lett. **69**, 2590 (1992)

Bernu, Lecheminant, Lhuillier, and Pierre, Phys. Rev. B **50**, 10048 (1994)



A family of states with S up to $O(\sqrt{N})$ collapse to the ground state with $\Delta E_S \propto S(S+1)/N$

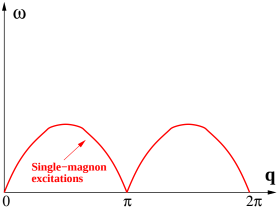
In the thermodynamic limit $\Delta E_S \to 0$
Linear combinations of states with different S $\Longrightarrow$ broken SU(2) symmetry

Inelastic Neutron scattering: magnon excitations and continuum

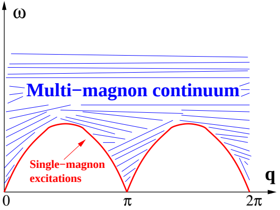
The inelastic Neutron scattering is a probe for the dynamical structure factor

$$S(q,\omega) = \int dt \langle \Psi_0 | S^{\alpha}_{-q}(t) S^{\alpha}_{q}(0) | \Psi_0 \rangle e^{i\omega t} = \sum_{n \neq 0} |\langle \Psi_n | S^{\alpha}_{q} | \Psi_0 \rangle|^2 \delta(\omega - \Delta\omega_{n0})$$

Within the harmonic approximation there is only a single branch of excitations (magnons)



In reality, a continuum of multi-magnon excitations exists above the threshold.
Single magnon excitations are well defined
$S(q,\omega) = Z_q \delta(\omega - \omega_q) +$ incoherent part



Mechanisms to destroy the long-range order

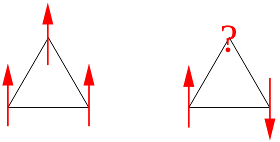
- Small value of the spin S, e.g., $S = 1/2$ or $S = 1$

  Stay away from the classical limit, e.g., large S

- Frustration of the super-exchange interactions

  Not all terms of the energy can be optimized simultaneously:

  Even at the classical level, many competing low-energy states



- Low spatial dimensionality: $D = 2$ is the "best" choice

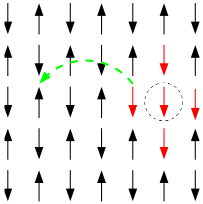
  Zero-point quantum fluctuations are huge

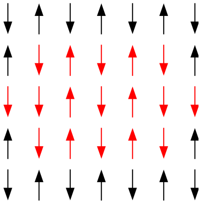
- Large continuous rotation symmetry group, e.g., SU(2), SU(N) or Sp(2N)

Arovas and Auerbach, Phys. Rev. B **38**, 316 (1988); Arovas and Auerbach, Phys. Rev. Lett. **61**, 617 (1988)

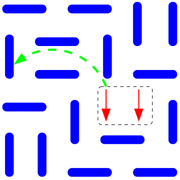
Read and Sachdev, Phys. Rev. Lett. **66**, 1773 (1991); Read and Sachdev, Nucl. Phys. **B316**, 609 (1989)

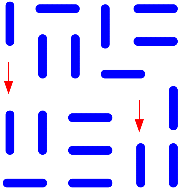
- Fractionalization is not compatible with magnetic order





- Fractionalization may exist in RVB liquids





A first definition for spin liquids

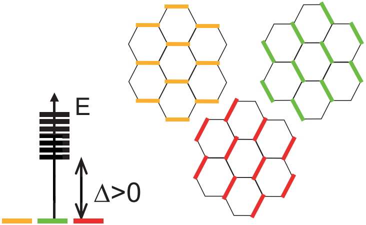
A spin liquid is a state without magnetic order

$$S(q) = \frac{1}{N} \langle \Psi_0 | \left| \sum_j \mathbf{S}_j e^{iqr_j} \right|^2 | \Psi_0 \rangle = \frac{1}{N} \sum_{j,k} \langle \Psi_0 | \mathbf{S}_j \cdot \mathbf{S}_k | \Psi_0 \rangle e^{iq(r_j - r_k)}$$

$$S(q) = \begin{cases} O(1) & \text{for all q's} \quad \rightarrow \text{short-range correlations} \\ S(q_0) \propto N & \text{for} q = q_0 \quad \rightarrow \text{long-range order} \end{cases}$$

The structure factor $S(q)$ does not diverge, whatever the q is

- It can be checked by using Neutron scattering
- The Mermin-Wagner theorem implies that *any* 2D Heisenberg model at $T > 0$ is a spin liquid according to this definition



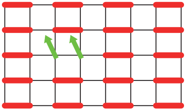
$= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ Singlet, total spin S=0

$J_1 - J_2$ Heisenberg model on the hexagonal lattice

Fouet, Sindzingre, and Lhuillier, Eur. Phys. J. B **20**, 241 (2001)

Properties:

- Short-range spin-spin correlations
- Spontaneous breakdown of some lattice symmetries $\rightarrow$ ground-state degeneracy
- Gapped $S = 1$ excitations ("magnons" or "triplons")



Spin liquid: a second definition

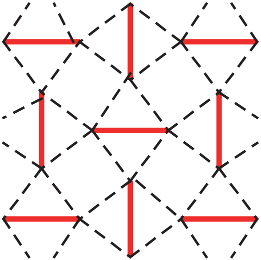
A spin liquid is a state without any spontaneously broken (local) symmetry

- It rules out magnetically ordered states that break spin SU(2) symmetry (including states with quadrupolar order)
- It rules out valence-bond crystals that break some lattice symmetries
- It rules out chiral spin liquids that break time-reversal symmetries

Wen, Wilczek, and Zee, Phys. Rev. B **39**, 11413 (1989)

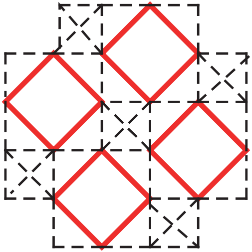
Notice: local means that there is a physical order parameter that can be measured by some local probe

There are few examples of magnetic insulators without any broken symmetry



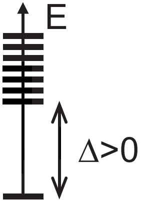
$SrCu_2(BO_3)_2$

Kageyama et al., Phys. Rev. Lett. **82**, 3168 (1999)



CaV_4O_9

Taniguchi et al., J. Phys. Soc. Jpn. **64**, 2758 (1995)

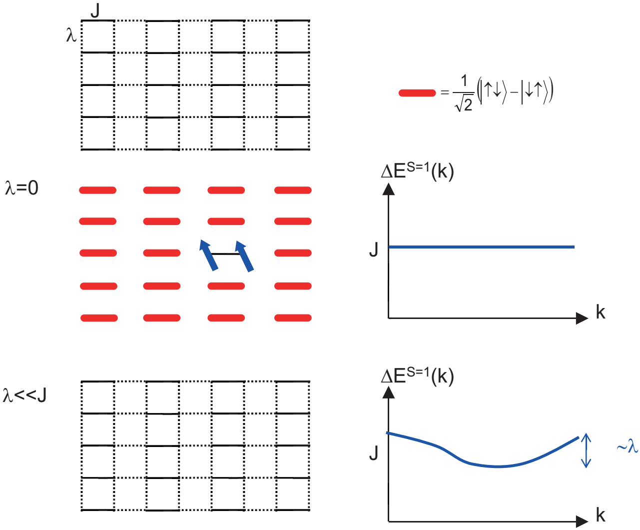


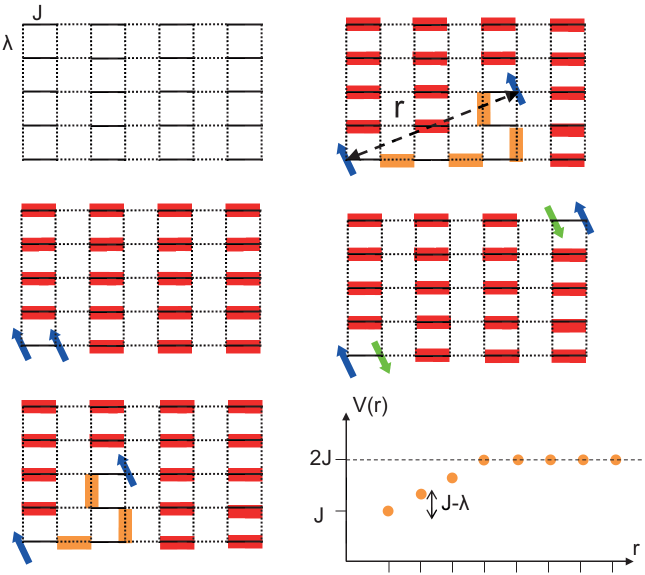
Non-degenerate ground state

Properties:

- No broken symmetries
- Even number of spin-1/2 in the unit cell
- Adiabatically connected to the (trivial) limit of decoupled blocks
- No phase transition between $T = 0$ and $\infty \Longrightarrow$ "simple" quantum paramagnet

Quantum paramagnets:excitation spectrum




J
λ
r
V(r)
2J
J
J-λ
r

A system with half-odd-integer spin in the unit cell cannot have a gap and a unique ground state

Valid in the thermodynamic limit for periodic boundary conditions and

$$L_1 \times L_2 \times \cdots L_D = odd$$

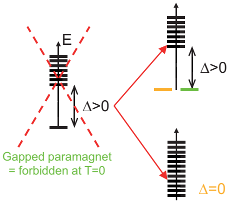
- The original theorem by Lieb, Schultz, and Mattis refers to 1D

Lieb, Schultz, Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961); Affleck and Lieb, Lett. Math. Phys. **12**, 57 (1986)

- Since then, several attempts to generalize it in 2D

Affleck, Phys. Rev. B **37**, 5186 (1988); Bonesteel, Phys. Rev. B **40**, 8954 (1989);

Oshikawa, Phys. Rev. Lett. **84**, 1535 (2000); Hastings, Phys. Rev. B **69**, 104431 (2004)



Case 1) Ground-state degeneracy
a) Valence-bond crystal
b) Resonating-valence bond spin liquid
(gapped but with a topological degeneracy)
Case 2) Gapless spectrum
a) Continuous broken symmetry (magnetic order)
b) Resonating-valence bond spin liquid
(gapless, i.e., critical state)

Implications of the LSM theorem

- Let us consider a two-dimensional system with $L_x \times L_y$ and L_y odd with periodic boundary conditions

- Let us consider the case with $S = 1/2$

If we assume that a gapped state is realized

then its ground state must be degenerate

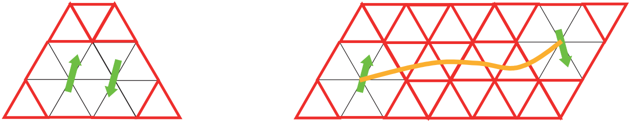
- Usually, ground-state degeneracies imply some spontaneous symmetry breaking
  There exists a local operator such that $\langle \Psi_1 | \mathcal{O} | \Psi_2 \rangle \neq 0$

- If the system does not break any symmetry, still the ground state must be degenerate
  The LSM degeneracy cannot be understood from a symmetry-breaking picture
  For all local operators $\langle \Psi_1 | \mathcal{O} | \Psi_2 \rangle = 0$

The degeneracy is related to **TOPOLOGICAL** properties $\Rightarrow$ **TOPOLOGICAL ORDER**

Wen, Phys. Rev. B **40**, 7387 (1989); Phys. Rev. B **44**, 2664 (1991)

- No dimer order $\rightarrow$ deconfined spinons:



- Spinon fractionalization and topological degeneracy go hand in hand

Ground states that are not connected by any local operator

Oshikawa and Senthil, Phys. Rev. Lett. **96**, 060601 (2006)

- A particularly insightful example is given by the Toric Code

Kitaev, Annals Phys. **303**, 2 (2003)

Spin liquid: a third definition

A spin liquid is a state without any spontaneously broken (local) symmetry, with a half-odd-integer spin in the unit cell

- It rules out magnetically ordered states that break spin SU(2) symmetry (including states with quadrupolar order)
- It rules out valence-bond crystals that break some lattice symmetries
- It rules out chiral spin liquids that break time-reversal symmetries

Wen, Wilczek, and Zee, Phys. Rev. B **39**, 11413 (1989)

- It rules out quantum paramagnets that have an even number of spin-half per unit cell

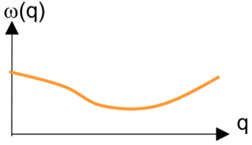
A spin liquid sustains fractional (spin-1/2) excitations

Inelastic Neutron scattering: spinon continuum

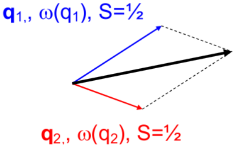
The inelastic Neutron scattering is a probe for the dynamical structure factor

$$S(q,\omega) = \int dt \langle \Psi_0 | S^{\alpha}_{-q}(t) S^{\alpha}_{q}(0) | \Psi_0 \rangle e^{i\omega t}$$

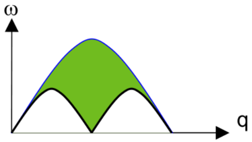
- The elementary excitations are spin-1 magnons: $S(q,\omega)$ has a single-particle pole at $\omega = \omega(q)$



- The spin-flip decays into two spin-1/2 excitations $S(q,\omega)$ exhibits a two-particle continuum

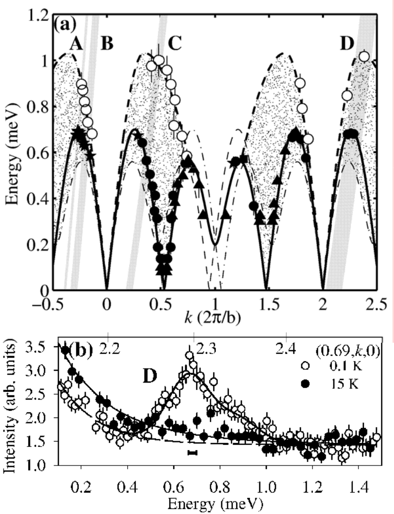


$q = q_1 + q_2$
$\omega = \omega(q_1) + \omega(q_2)$
S=0 or 1



Neutron scattering on Cs_2CuCl_4

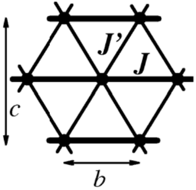
Coldea, Tennant, Tsvelik, and Tylczynski , Phys. Rev. Lett. **86**, 1335 (2001)



Almost decoupled layers

Strongly-anisotropic triangular lattice

$J' \simeq 0.33J$: **quasi-1D**



Proof of the Lieb-Shultz-Mattis theorem in 1D

- Consider the Heisenberg model on a chain:

$$\mathcal{H} = \sum_{i=1}^{N} \mathbf{S}_i \cdot \mathbf{S}_{i+1}$$

with periodic boundary conditions ($\mathbf{S}_{N+1} \equiv \mathbf{S}_1$), even N, and half-odd integer spins

Theorem:

There exists an excited state with an energy that vanishes as $N \to \infty$

- $|\Psi_0\rangle$ is the ground state of $\mathcal{H}$ with energy E_0.
- Assume that $|\Psi_0\rangle$ is a singlet ("almost" always the case)
- Consider the twist operator $\mathcal{O} = \exp\{\frac{2\pi i}{N} \sum_{j=1}^{N} j S_j^z\}$
- Denote $|\Psi_1\rangle = \mathcal{O}|\Psi_0\rangle$

Then:

(1) $\langle\Psi_1|\Psi_0\rangle = 0$
(2) $\lim_{N\to\infty}[\langle\Psi_1|\mathcal{H}|\Psi_1\rangle - E_0] = 0$

Proof of the Lieb-Shultz-Mattis theorem in 1D

Consider the translation operator $\mathcal{T}$:

$$\mathcal{T}\mathbf{S}_j\mathcal{T}^{-1} = \mathbf{S}_{j+1} \qquad \mathcal{T}\mathbf{S}_N\mathcal{T}^{-1} = \mathbf{S}_1$$

$$[\mathcal{H},\mathcal{T}] = 0 \qquad \mathcal{T}|\Psi_0\rangle = e^{ik_0}|\Psi_0\rangle$$

$\langle\Psi_0|\Psi_1\rangle = \langle\Psi_0|\mathcal{O}|\Psi_0\rangle = \langle\Psi_0|\mathcal{T}\mathcal{O}\mathcal{T}^{-1}|\Psi_0\rangle$

$\mathcal{T}\mathcal{O}\mathcal{T}^{-1} = \mathcal{O}\exp\left(2\pi i S_1^z\right)\exp\left(-\frac{2\pi i}{N}S^z_{\mathrm{tot}}\right)$

Then, $\exp\left(-\frac{2\pi i}{N}S^z_{\mathrm{tot}}\right)|\Psi_0\rangle = |\Psi_0\rangle$, since $|\Psi_0\rangle$ is a singlet.

$$\exp\left(2\pi i S_1^z\right) = \begin{cases} +1 & S = 0,1,2,\cdots \\ -1 & S = 1/2, 3/2, 5/2, \cdots \end{cases}$$

- Therefore, for half-odd integer spin: $\langle\Psi_0|\Psi_1\rangle = -\langle\Psi_0|\Psi_1\rangle$

$\langle\Psi_1|\mathcal{H}|\Psi_1\rangle = E_0 + \langle\Psi_0|\{\cos(\frac{2\pi}{N}) - 1\}\sum_{j=1}^{N}(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y)|\Psi_0\rangle$

$\langle\Psi_0|(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y)|\Psi_0\rangle \le S^2$

- We obtain an upper-bound for the energy: $\langle\Psi_1|\mathcal{H}|\Psi_1\rangle - E_0 \le \frac{2\pi^2 J S^2}{N} + O(N^{-3})$

Generalization by Affleck in 2D

- The previous argument can be easily generalized for odd L_y

  (odd-leg ladders $L_x \times L_y$)

  Affleck, Phys. Rev. B **37**, 5186 (1988)

  As before, it is possible to show that:

  $\langle \Psi_0 | \Psi_1 \rangle = -\langle \Psi_0 | \Psi_1 \rangle$

  $\langle \Psi_1 | \mathcal{H} | \Psi_1 \rangle - E_0 \leq \frac{2\pi^2 J S^2 L_y}{L_x}$

It works whenever $L_x \gg L_y$

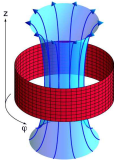
Generalization by Oshikawa in 2D

- A further generalization that does not require a strong anisotropy is possible (Related to the Laughlin argument for the the quantum Hall effect)

Laughlin, Phys. Rev. B **23**, 5632 (1981)

- Insert a twist in the spin Hamiltonian

$$\mathcal{H}(\theta) = \sum_{m,n} J_{m,n} S^z_m S^z_n + \frac{1}{2} \sum_{m,n} J_{m,n} e^{i\theta(x_m - x_n)/L_x} S^+_m S^-_n + h.c.$$



- The spectra of $\mathcal{H}(2\pi)$ and $\mathcal{H}(0)$ are the same: $\mathcal{U}\mathcal{H}(0)\mathcal{U}^{-1} = \mathcal{H}(2\pi)$

$$\mathcal{U} = \prod_m \exp\left(2\pi i \frac{x_m}{L_x} S^z_m\right)$$

$$|\Psi_0(2\pi)\rangle = \mathcal{U}|\Psi_0(0)\rangle$$

- The operator $\mathcal{U}$ does not commute with translations along x

$$\mathcal{T}\mathcal{U}\mathcal{T}^{-1} = \mathcal{U} \exp\left(2\pi i L_y S\right) \exp\left(-\frac{2\pi i}{L_x} S^z_{\text{tot}}\right)$$

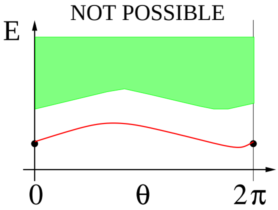
Generalization by Oshikawa in 2D

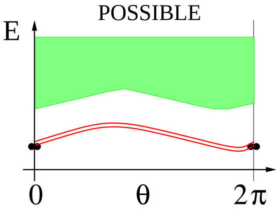
- Given a gapped $\mathcal{H}(0)$, assume that $\mathcal{H}(\theta)$ remains gapped for all fluxes $0 \leq \theta \leq 2\pi$
- Imagine that we adiabatically insert a flux θ from 0 to 2π

  Then $|\Psi_0(0)\rangle$ evolves into $|\Psi(2\pi)\rangle = \mathcal{U}|\Psi_0(0)\rangle$

  However, whenever $L_y S$ is an half-odd integer (e.g., L_y odd and $S = 1/2$) $|\Psi_0(0)\rangle$ and $\mathcal{U}|\Psi_0(0)\rangle$ have different quantum numbers

Therefore, the ground state must be degenerate





It works whenever L_y is odd

On finite systems, a small gap between these states is possible

Oshikawa, Phys. Rev. Lett. **84**, 1535 (2000)