Chapter 19

Lecture 19

19.1 Properties of Einstein's field equation

We are now going to discuss some properties of Einstein equations. We derived them *in vacuo* and we are now interested in analyzing better their structure. In particular (we already underlined this in the previous lecture) it is evident that Einstein equations *in vacuo*, which we rewrite here as[1]

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = 0$$

are a system of ten non-linear partial differential equations of the second order for the 10 unknown functions $g_{\mu\nu}$, i.e. for the metric tensor. The Ricci tensor comes from the Riemann tensor

$$R^\alpha{}_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\beta\rho}\Gamma^\rho_{\nu\mu} - \Gamma^\alpha_{\nu\rho}\Gamma^\rho_{\beta\mu}$$

through a contraction., where we also remember that

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}\left(-\partial_\mu g_{\beta\gamma} + \partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\mu\beta}\right).$$

Note that, from the two previous expression, the non-linearity of the equations is quite evident, especially because of the terms containing the products of the connection symbols in the Riemann tensor. Let us now carefully consider the first two terms, where second derivatives of the metric tensor appear. We rewrite these two terms, using the anti-symmetrization square brackets, as

$$\partial_{[\beta}\Gamma^\alpha_{\nu]\mu}$$

and substitute inside the expression for the connection symbols, which become

$$\begin{aligned}
\partial_{[\beta}\Gamma^\alpha_{\nu]\mu} &= \frac{1}{2}\partial_{[\beta}\left[g^{\alpha\rho}\left(-\partial_\rho g_{\nu]\mu} + \partial_{\nu]} g_{\mu\rho} + \partial_\mu g_{\rho\nu]}\right)\right] \\
&= [\dots] + \frac{1}{2} g^{\alpha\rho}\left(-\partial_{[\beta}\partial_\rho g_{\nu]\mu} + \partial_{[\beta}\partial_{\nu]} g_{\mu\rho} + \partial_{[\beta}\partial_\mu g_{\rho\nu]}\right),
\end{aligned}$$

[1] Note that we do not pay too much attention to which of the indices is the first and which the second, since the expression is symmetric!

so that

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial^2_{\nu\rho} g_{\mu\sigma} + \partial^2_{\mu\sigma} g_{\nu\rho} - \partial^2_{\nu\sigma} g_{\mu\rho} - \partial^2_{\mu\rho} g_{\nu\sigma}\right) + g_{\alpha\beta}\left(\Gamma^\alpha_{\rho\nu}\Gamma^\beta_{\sigma\mu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\beta_{\rho\mu}\right).$$

How can the above expression contribute a second derivative with respect to x^0 of one of the components $g_{0\mu}$? At least we need three indices being 0, but remembering the symmetry properties of the Riemann tensor, the corresponding component is then zero! Thus the Riemann tensor contains second derivatives with respect to x^0 of the components

$$g_{ij} \quad i = 1, 2, 3, \quad j = 1, 2, 3$$

of the metric tensor, **but it does not** contain second derivatives with respect to x^0 of the components $g_{0\mu}$ of the metric tensor! The same is true of the Ricci tensor (contraction is just an algebraic operation) and of the Ricci scalar. Thus the Einstein tensor contains second derivatives with respect to x^0 only of the g_{ij} **and not** of the $g_{0\mu}$. Thus in the Einstein equations only second derivatives with respect to x^0 of the g_{ij} appear!

Let us now analyze this situation more carefully. In particular, remember that the Einstein tensor, $G_{\mu\nu}$ obeys the Bianchi identities, $G^\mu_{\nu;\mu} = 0$. This relation can be rewritten as

$$\left(R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R\right)_{;\mu} = 0$$

or, which is the same,

$$\left(R^0_\nu - \frac{1}{2}\delta^0_\nu R\right)_{;0} = -\left(R^i_\nu - \frac{1}{2}\delta^i_\nu R\right)_{;i}.$$

Now, in the right-hand side we have at most second derivatives with respect to x^0. On the left-hand side then we can thus have at most first derivatives with respect to x^0 in the expression

$$R^0_\nu - \frac{1}{2}\delta^0_\nu R.$$

Moreover this expression cannot contain derivatives with respect to x^0 of the quantities $g_{0\mu}$. These derivatives require two zero indices in the Riemann tensor, i.e. they appear in components of the type R_{0i0j}. But this components cannot appear in the Einstein equation with one index equal to 0. We can thus split the full set of Einstein equations in two subsets[2]:

1. A first one containing the six equations with *spatial* indices:

   $$R^i_j - \frac{1}{2}\delta^i_j R = 0.$$

   These equations contain second derivatives with respect to x^0 of the g_{ij} and first derivatives with respect to x^0 of the $g_{0\mu}$.

[2]Remember that Latin indices cover the *spatial* values 1, 2, 3, whereas Greek indices cover the *spacetime* values 0, 1, 2, and 3.

2. A second one containing the remaining four equations where at least one index is a *temporal* index:
$$R^0_j = 0 \quad \text{and} \quad R^0_0 - \frac{1}{2}R = 0.$$
These equations contain first derivatives with respect to x^0 of the g_{ij}, no second derivatives with respect to x^0 and no first derivatives with respect to x^0 of the $g_{0\mu}$.

In light of the above observations, what are then a proper set of initial conditions to solve Einstein equations with respect to x^0? Certainly we cannot assign both the 10 metric fields and their derivatives, since in general such a set of initial value data will not satisfy Einstein equations at the starting time. We can of course give the six metric fields g_{ij} and their derivatives $\partial_0 g_{ij}$. From these initial values we can then determine the initial values for the fields $g_{0\mu}$, using the set of equation *2.* above. We see that in this sense (and also from their differential character) the set of equations *2.* acts as a **constraint** on the system. The quantities $g_{0\mu}$ are not completely determined by the initial conditions and the evolution: what is their physical meaning? They represent the *arbitrarines in the choice of the reference system* as a result of the principle of general covariance. This is a crucial evidence of the interplay between gravitation and the choice of the reference system, i.e. of the interplay between general covariance, the equivalence principle and the theory of gravity which we discussed in lecture 16. After having determined also the $g_{0\mu}$ at the initial time, we have a complete consistent set of initial data, from which we can start to solve the equations. But note that at each *tick of our time* we have some freedom in determining the $g_{0\mu}$, which are not fully dynamically determined.

19.2 Physical meaning of the metric fields

We are now interested, also in light of the discussion in the above section, to discuss in more detail the physical meaning of the metric fields. As a short preliminary summary, we will see how the metric fields are related to the measurements of time intervals and of spatial distances.

19.2.1 Time measurements

Let us consider two events, at the same specetime point and the invariant interval ds^2 will be just $-c^2 d\tau^2$, where $d\tau$ is the time interval measured by a clock that occupies the same space position $dx^i = 0$. Thus we have
$$-c^2 d\tau^2 = ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00}(dx^0)^2.$$
We will call τ the proper time at a given point in space. From the above we get
$$d\tau = \frac{\sqrt{-g_{00}}}{c} dx^0.$$
g_{00} must then be negative. If a metric tensor of a Lorentzian manifold does not satisfy this condition, this means that it describes a reference system that cannot be realized by physical bodies. Thus with an appropriate change of the reference system (i.e. passing to a realizable one) we can always fulfill the condition $g_{00} < 0$.

19.2.2 Space measurements

A slightly more complicated treatment is required for space measurements. This is strictly related to the property, we just discussed, that time flows in a different way at different points in space when a gravitational field is present. Let us thus imagine the following setup:

> A light signal is emitted from an *event* m $\in \mathscr{M}$ toward an event p $\in \mathscr{M}$ and reflected from p back to m. Let us assume the spatial coordinates at m are $x^i + dx^i$, whereas at p they are x^i. Moreover, *the time required for the process as seen by* m *is* c *times twice the spatial distance between* m *and* P.

Let us write

$$ds^2 = g_{ij} dx^i dx^j + 2g_{0i} dx^0 dx^i + g_{00} dx^0 dx^0.$$

At the same time, since we are using light lays, i.e. null geodesics,

$$ds^2 = 0$$

or equivalently,

$$[g_{00}](dx^0)^2 + [2g_{0i}dx^i](dx^0) + [g_{ij}dx^i dx^j] = 0, \tag{19.1}$$

so that from the above we find two values for dx^0 for the given spatial separation dx^i, let us call them

$$(dx^0)_{(-)} \quad \text{and} \quad (dx^0)_{(+)}.$$

This values correspond to the emission and arrival of the light signal at m. The *coordinate* time interval is thus

$$\Delta^0 = |(dx^0)_{(+)} - (dx^0)_{(-)}|$$

so that the corresponding time interval, using the result above, is

$$\Delta\tau = \frac{\sqrt{-g_{00}}}{c}\Delta^0 = \frac{\sqrt{-g_{00}}}{c}\left|(dx^0)_{(+)} - (dx^0)_{(-)}\right|$$

and the spatial distance

$$\Delta l = \frac{c}{2}\Delta\tau = \frac{\sqrt{-g_{00}}}{2}\Delta^0 = \frac{\sqrt{-g_{00}}}{2}\left|(dx^0)_{(+)} - (dx^0)_{(-)}\right|.$$

But the difference $|(dx^0)_{(+)} - (dx^0)_{(-)}|$ is twice the absolute value of the discriminant of the second order equation (19.1) divided by $2|g_{00}|$, so that

$$\Delta l = \sqrt{\frac{(g_{00}g_{ij} - g_{0i}g_{0j})dx^i dx^j}{g_{00}}}.$$

This result gives the spatial separation in terms of the spatial change of coordinates, according to the definition at the beginning of this subsection. It is crucial to note that all the quantities appearing in the expression for Δl depend also on the coordinate x^0: thus the expression has a purely *local* meaning.

19.2.3 Clock synchronization

Let us now consider the problem of synchronization of clocks. We can use the same set up we used above, with the two events `m` and `p`. According to this situation we will consider simultaneous with `p` the event on the worldline passing through `m` which corresponds to

$$x^0 + \Delta x^0 = x^0 + \frac{1}{2}\left((dx^0)_{(+)} + (dx^0)_{(-)}\right).$$

The sum of $(dx^0)_{(+)}$ and $(dx^0)_{(-)}$ is given by the opposite of the coefficient of the linear term in dx^0 in equation (19.1) divided by g_{00}, so we obtain

$$\Delta x^0 = -\frac{g_{0i}dx^i}{g_{00}}.$$

Using this result we can always synchronize clocks along a non-closed line. But synchronization is in general globally impossible. It becomes possible only when all the g_{0i} are equal to zero. From what we have seen above, studying the structure of Einstein equations, this property of synchronization is a result of the choice of the reference system and not of the gravitational field itself. To show this we are going to give an *intuitive* idea of how it is always possible to (*locally*) choose a reference system such that the components g_{0i} of the metric vanish (and $g_{00} = 1$). This system will be called a *synchronous reference system.* The construction goes as follows.

> Choose a spacelike hypersurface, i.e. an hypersurface whose normal is always a timelike vector. Starting from this hypersurface we can always construct a family of geodesics that are normal to this hypersurface. Moreover this geodesics are timelike. We are going to call the proper length along one of these geodesics the temporal coordinates, whereas the three numbers identifying a geodesic in the family constitute the spatial coordinates.

By construction, since the geodesics are normal to the initial hypersurface, the g_{0i} are zero. Moreover, using the proper length along the geodesic, we can make sure that $g_{00} = 1$.

19.3 More about the classical limit

Let us now consider again the situation we reached at the end of Lecture 17, when we have written equation (17.3). That equation

$$\ddot{x}^k = -\Gamma^k_{00}$$

was what we called the *classical limit* of the geodesic equation. It was expressing that from the point of view of the Newtonian theory, a force per unit mass Γ^k_{00} was acting on a test particle (following a geodesic). Now, after writing Einstein equations, we have more clear a relation between the metric, the choice of a reference system and the gravitational field. Now that we have at hand Einstein equations we can give more substance to this observation.

First of all let us consider the 00-component of Einstein equations *in vacuo*. It is $R_{00} = 0$. But R_{00} is given by

$$R_{00} = \partial_\rho \Gamma^\rho_{00} - \partial_0 \Gamma^\rho_{\rho 0} + \Gamma^\rho_{\rho\beta}\Gamma^\beta_{00} - \Gamma^\beta_{0\rho}\Gamma^\rho_{\beta 0}.$$

Neglecting time derivatives compared to spatial derivatives and all quadratic terms, we obtain

$$R_{00} \approx \partial_k \Gamma^k_{00}.$$

We are now going to discuss in more detail exactly the expression of Γ^k_{00},

$$\Gamma^k_{00} = \frac{1}{2} g^{k\mu} \left(-\partial_\mu g_{00} + \partial_0 g_{0\mu} + \partial_0 g_{\mu 0} \right).$$

Choosing a synchronous reference system and neglecting derivative with respect to x_0 we then obtain

$$\Gamma^k_{00} = -\partial^k g_{00}.$$

The geodesic equation and Einstein equation with indices 00, are thus simplified to:

$$\begin{aligned} \ddot{x}^k &= -\partial^k g_{00} \\ \partial_k \partial^k g_{00} &= 0. \end{aligned}$$

Thus the geodesic particle is acted upon by a force per unit mass which is minus the gradient of the potential g_{00}. In vacuum this potential satisfies Poisson equation: it is the analogous, in our limit, of the Newtonian potential. In particular, as we will see later on, we will have the identification

$$-g_{00} = 1 + \frac{2\phi}{c^2},$$

where ϕ is exactly the Newtonian gravitational potential.

19.4 Synopsis

We have described a framework for the description of gravitation, in the sense that it is a proper generalization of Newtonian gravitation. The gravitational potential is now codified by the geometrical (metric) structure of the spacetime (manifold). The action for the gravitational field is the only invariant that can be constructed from the curvature tensor and that gives a non-trivial dynamics. Thanks to the fact that in this invariant second derivatives of the fields (gravitational potentials) appear linearly, we obtain a consistent set of second order differential equations. Note that we have also a precise physical interpretation of why a Lagrangian density for a gravitational field which is a scalar cannot be obtained with only the metric field and their first derivative (the connection symbols): in fact, because of the general covariance of the theory, we can always choose the metric to obtain a connection which vanishes at a point in space; thus the only scalars that can be constructed with $g_{\mu\nu}$ and $\Gamma^\alpha_{\beta\gamma}$ are constants. To obtain a non-constant scalar we need at least second derivatives of the metric tensor: they can have a coordinate independent meaning (curvature) and the Ricci scalar is the simplest of them.

Our next task is to free ourselves from being in vacuum, i.e. to discuss how sources of the gravitational field can be defined and what is their relationship with the gravitational field and the geometry of spacetime.