Chapter 18

Lecture 18

18.1 Einstein Equations

In the previous lecture we presented the mathematical structure of General Relativity, but we omitted actually the most important part. We gave some ideas of how a manifold structure naturally embodies some properties that can be seen as natural mathematical translations of the physical principles underlying the theory, as we learned them from Einstein's words in lecture 16. We also quickly said, passing by, that the curvature of the manifold embodies somehow the gravitational field, but without adding any precise or rigorous meaning to our words. So we are still far from our ultimate goal: "find some quantities that adequately describe the gravitational field and determine the equations to which these quantities, i.e. the field, are subjected to".

18.1.1 Action for the gravitational field

We are thus going to assume that the metric tensor is the collection of 10 fields suitable for our newly (and still to come) description of the gravitational field. According to some principles we laid down in lecture 3 we would like to derive some equations for the gravitational field starting from a variational principle. Mathematically this means we need to find a scalar function of the fields and of their first derivatives to use as a Lagrangian density. Then we can construct the action, i.e. our functional integral of the fields, and perform a variation to obtain the field equations.

18.1.2 Variational principle

18.1.3 The Lagrangian density

We will take as a Lagrangian density the Ricci scalar R: this is the simplest scalar quantity that we can construct with the metric, the first derivatives of the metric and the second derivatives of the metric, in the sense that it is linear in the second derivatives. No other nontrivial scalar quantities can be built from the curvature tensor through operations that preserve the linearity in the second derivatives. With apologies if this assumption sounds rather unjustified at this time, we ask you to follow us in a (not so straightforward) calculation that starts by taking

$$\mathcal{L}(\boldsymbol{g}, \partial \boldsymbol{g}) = R = g^{\mu\nu} R_{\mu\nu}$$

as a Lagrangian theory for a description of the gravitational field[1].

18.1.4 Derivations of Einstein equations in vacuo

Let us thus tart with an action which is the spacetime integral of the Lagrangian density R,

$$\mathcal{S}_{\mathrm{G}}[\boldsymbol{g}] = \int_{\mathscr{M}} \mathcal{L}(\boldsymbol{g}, \partial \boldsymbol{g}) \sqrt{-g} d^4 x = \int_{\mathscr{M}} R \sqrt{-g} d^4 x,$$

where we remember that the Ricci scalar is given by

$$R = g^{\mu\nu} R_{\mu\nu}.$$

We will explicitly find the field equations for $g^{\mu\nu}$ associated to the above Lagrangian density by computing the variation of the action

$$\delta \mathcal{S}_{\mathrm{G}}[\boldsymbol{g}] = \int_{\mathscr{M}} d^4 x \left[\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} (\delta g_{\mu\nu}) R^{\mu\nu} + g^{\mu\nu} R_{\mu\nu} \delta(\sqrt{-g}) \right]. \quad (18.1)$$

From equation (13.1) contracted, that is reported here in the notation we are using now

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\alpha\beta} \Gamma^\beta_{\nu\mu} - \Gamma^\alpha_{\nu\beta} \Gamma^\beta_{\alpha\mu},$$

we see that $R_{\mu\nu}$ contains our fields $g_{\mu\nu}$, their first derivatives (since $\Gamma^\alpha_{\beta\gamma}$ contain derivatives of $g_{\rho\sigma}$) as well as the second derivatives, since also the derivatives of Γ's appear in the Riemann tensor, of which $R_{\mu\nu}$ is the contraction. From the preliminary variation (18.1) we see that the second term in square brackets already has is in the final form in which is useful for our computation. A little elaboration is needed instead for the first and the last terms.

Let us start with the last one. We have to compute the variation of the determinant of the metric, $\sqrt{-g}$. To this end we write the determinant as the exponential of the logarithm of the determinant. We also remember that given a matrix its logarithm and its exponential are defined in terms of the corresponding power series (using matrix product). Then we have:

$$\begin{aligned}
\delta g &= \delta(\det(\boldsymbol{g})) \\
&= \delta(\exp(\log(\det(\boldsymbol{g})))) \\
&= \delta(\exp(\mathrm{Tr}(\log(\boldsymbol{g})))) \\
&= \exp(\mathrm{Tr}(\log(\boldsymbol{g})) \cdot \delta(\mathrm{Tr}(\log(g_{\mu\nu})) \\
&= \exp(\log(\det(\boldsymbol{g})) \cdot \delta(\mathrm{Tr}(\log(g_{\mu\nu})) \\
&= g \cdot \mathrm{Tr}(\delta(\log(g_{\mu\nu})) \\
&= g \cdot \mathrm{Tr}(g^{\alpha\mu} \delta g_{\mu\nu}) \\
&= g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu},
\end{aligned} \quad (18.2)$$

where we remember that

$$g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha_\beta$$

[1]If this discussion still does not satisfy you and you think that the following complications are too much effort to be undertaken blindly, please consider that this effort will bring you to know Einstein equations *in vacuo*.

so that

$$(\delta g^{\alpha\mu}) g_{\mu\beta} + g^{\alpha\mu} \delta g_{\mu\beta} = 0,$$

i.e.

$$(\delta g^{\alpha\mu}) g_{\mu\beta} = -g^{\alpha\mu} \delta g_{\mu\beta}.$$

We now turn to the first term, in which the variation of the Ricci tensor appears, $\delta R_{\mu\nu}$. We remember that the components of the Riemann tensor in a given coordinate system are

$$R^{\alpha}{}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\nu\mu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\beta\rho}\Gamma^{\rho}_{\nu\mu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\beta\mu}$$

so that the Ricci tensor can be written as

$$R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\nu\mu} - \partial_\nu \Gamma^{\alpha}_{\alpha\mu} + \Gamma^{\alpha}_{\alpha\rho}\Gamma^{\rho}_{\nu\mu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\alpha\mu}.$$

We are now going to write the above expression in a locally inertial coordinate system, i.e. a coordinate system centered at a point and defined by the condition that at the point the connection symbols vanish. In this coordinate system (where we are going to denote tensor components with a "$\hat{\ }$") at the point x we thus have $\hat{\Gamma}^{\mu}_{\alpha\beta} \equiv 0$, although, of course in general $\hat{\partial}_\nu \hat{\Gamma}^{\mu}_{\alpha\beta} \neq 0$. In this coordinate system the expression of the Ricci tensor evidently simplifies and becomes

$$\hat{R}_{\mu\nu} = \hat{\partial}_\alpha \hat{\Gamma}^{\alpha}_{\nu\mu} - \hat{\partial}_\nu \hat{\Gamma}^{\alpha}_{\alpha\mu}.$$

We are now going to perform the variation in this coordinates system and return to the previous one at the end. In particular we remember that, *although the connection symbols are* **not** *a tensor*, differences between a connection symbols are a tensor. In general the variation of the connection symbols is a difference between two of them, so $\delta\Gamma^{\mu}_{\alpha\beta}$ are the components of a tensor. Thus when we write

$$\begin{aligned}
\delta \hat{R}_{\mu\nu} &= \delta(\hat{\partial}_\alpha \hat{\Gamma}^{\alpha}_{\nu\mu}) - \delta(\hat{\partial}_\nu \hat{\Gamma}^{\alpha}_{\alpha\mu}) \\
&= \hat{\partial}_\alpha(\delta\hat{\Gamma}^{\alpha}_{\nu\mu}) - \hat{\partial}_\nu(\delta\hat{\Gamma}^{\alpha}_{\alpha\mu}) \\
&= (\delta\hat{\Gamma}^{\alpha}_{\nu\mu})_{\hat{,}\alpha} - (\delta\hat{\Gamma}^{\alpha}_{\alpha\mu})_{\hat{,}\nu} \\
&= (\delta\hat{\Gamma}^{\alpha}_{\nu\mu})_{\hat{;}\alpha} - (\delta\hat{\Gamma}^{\alpha}_{\alpha\mu})_{\hat{;}\nu},
\end{aligned}$$

all quantities appearing on the above are tensors. As a tensor equation the result is valid in any coordinate system, not only in the locally inertial one that we have chosen, so that in full generality we can write

$$\delta R_{\mu\nu} = (\delta\Gamma^{\alpha}_{\nu\mu})_{;\alpha} - (\delta\Gamma^{\alpha}_{\alpha\mu})_{;\nu}. \tag{18.3}$$

Let us look now at the combination $g^{\mu\nu}\delta R_{\mu\nu}$:

$$\begin{aligned}
g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}(\delta\Gamma^{\alpha}_{\nu\mu})_{;\alpha} - g^{\mu\nu}(\delta\Gamma^{\alpha}_{\alpha\mu})_{;\nu} \\
&= (g^{\mu\nu}\delta\Gamma^{\alpha}_{\nu\mu})_{;\alpha} - (g^{\mu\nu}\delta\Gamma^{\alpha}_{\alpha\mu})_{;\nu} \\
&= (g^{\mu\nu}\delta\Gamma^{\alpha}_{\nu\mu})_{;\alpha} - (g^{\mu\alpha}\delta\Gamma^{\nu}_{\nu\mu})_{;\alpha} \\
&= (g^{\mu\nu}\delta\Gamma^{\alpha}_{\nu\mu})_{;\alpha} - (g^{\mu\alpha}\delta\Gamma^{\nu}_{\nu\mu})_{;\alpha} \\
&= \left(g^{\mu\nu}\delta\Gamma^{\alpha}_{\nu\mu} - g^{\mu\alpha}\delta\Gamma^{\nu}_{\nu\mu}\right)_{;\alpha} \\
&= K^{\alpha}{}_{;\alpha},
\end{aligned}$$

where we set

$$K^{\alpha} = \left(g^{\mu\nu}\delta\Gamma^{\alpha}_{\nu\mu} - g^{\mu\alpha}\delta\Gamma^{\nu}_{\nu\mu}\right).$$

We thus have dealt with the first and last term. The second one is already in a convenient form and we can put results (18.2)[2] and (18.3) into (18.1) to obtain:

$$\begin{aligned}
\delta\mathcal{S}_{\mathrm{G}}[\boldsymbol{g}] &= \int_{\mathscr{M}} d^4x \left[\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} + \sqrt{-g}(\delta g_{\mu\nu})R^{\mu\nu} - \frac{(-g)g_{\mu\nu}}{2\sqrt{-g}}g^{\alpha\beta}R_{\alpha\beta}\delta g^{\mu\nu}\right] \\
&= \int_{\mathscr{M}} d^4x \left[\sqrt{-g}K^{\alpha}{}_{;\alpha} + \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}\right] \\
&= \int_{\mathscr{M}} d^4x\sqrt{-g}K^{\alpha}{}_{;\alpha} + \int_{\mathscr{M}} d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} \\
&= \int_{\partial\mathscr{M}} dV^{(3)}K^{\alpha}d\Sigma_{\alpha} + \int_{\mathscr{M}} d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}. \qquad (18.4)
\end{aligned}$$

Let us consider the first term. It is computed on the boundary $\partial\mathscr{M}$, where field variations (and their derivatives), are chosen to vanish (in a way similar to what we saw in lecture 3, when deriving the Euler-Lagrange equations (3.2)). Thus only the second term remains and the stationarity of the action

$$\delta\mathcal{S}_{\mathrm{G}}[\boldsymbol{g}] = 0$$

for all variations $\delta g^{\mu\nu}$ implies the equations of motions for the fields $g^{\mu\nu}(x)$. These are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

or equivalently

$$G_{\mu\nu} = 0.$$

These are *Einstein equations in vacuo.* Since $G_{\mu\nu}$ is symmetric they are a system of 10 non-linear partial differential equations of the second order for the 10 fields $g_{\mu\nu}(x)$.

[2]Remembering that

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}\delta(-g).$$