Chapter 16

Lecture 16

16.1 Notation, Greek indices, ...

We are going now to set up some notation for vectors, covectors and tensors that is commonly used in General Relativity. As an exercise, it is suggested to rewrite the results in the previous lectures about differential geometry with the following notation.

1. Greek indices μ, ν, ρ, σ, α, β, γ, ... (four dimensional or spacetime indices) always take the values 0, 1, 2, 3, unless something else is explicitly stated or indicated by an explicit "$\sum$" sign.

2. Latin indices i, j, k, l, a, b, c, ... (three dimensional or space indices) always take the values 1, 2, 3, unless something else is explicitly stated or indicated by an explicit "$\sum$" sign.

3. For all summation on Greek and Latin indices we are going to suppress the "$\sum$" sign, i.e. we adhere to Einstein convention on repeated indices: repeated indices (one of them *up* and the other *down*) always implicitly understand a summation on the values 0, 1, 2, 3 if they are Greek and on the values 1, 2, 3 if they are Latin. This means that in place of
$$\sum_{\alpha,\beta}^{0,3} \Gamma^{\mu}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}$$
we are going to write
$$\Gamma^{\mu}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}.$$
Similarly, in place of
$$\sum_{\alpha}^{0,3} R^{\alpha}{}_{\mu\alpha\nu} = R_{\mu\nu}$$
we are going to write
$$R^{\alpha}{}_{\mu\alpha\nu} = R_{\mu\nu}.$$

4. We are going to use the symbol $\nabla_{\mu}-$ interchangeably with the notation $-_{;\mu}$ to express the covariant derivative with respect to the unique symmetric connection compatible with the Riemannian metric; i.e.
$$\nabla_{\mu}\nabla_{\nu}Z^{\alpha} = Z^{\alpha}{}_{;\nu\mu}.$$

5. In a given coordinate system we are going to indicate the metric with $g_{\mu\nu}$. Thus given two vectors $\boldsymbol{V}$ and $\boldsymbol{W}$, with components v^μ and w^ν respectively, we are going to write

$$\langle \boldsymbol{V}, \boldsymbol{W} \rangle = g_{\mu\nu} v^\mu w^\nu$$

for their scalar product.

6. As we quickly discussed in section 9.1.1 the metric naturally gives an isomorphism between vectors and covectors. This isomorphisms (that can be naturally extended to the tangent and cotangent bundle of a manifold) associates to the vector $\boldsymbol{w} \in \mathscr{M}_{\mathtt{m}}$ the covector $\boldsymbol{\omega} \in \mathscr{M}^*_{\mathtt{m}}$ defined as

$$\boldsymbol{\omega}(-) = \langle \boldsymbol{w}, - \rangle .$$

Let us consider a vector $\boldsymbol{v}$. The action of $\boldsymbol{\omega}$ on $\boldsymbol{v}$ in terms of the respective components in a chosen basis (and dual basis) is given by

$$\boldsymbol{\omega}(\boldsymbol{v}) = \omega_\mu v^\mu .$$

On the other hand in the same basis (and dual basis)

$$\langle \boldsymbol{w}, \boldsymbol{v} \rangle = g_{\nu\mu} w^\nu v^\mu .$$

From the last three equations we obtain that in a chosen basis (and dual basis) the components of the covector $\boldsymbol{\omega}$ associated to the vector $\boldsymbol{w}$ by the isomorphism induced by the metric satisfy the following identity[1]:

$$\omega_\mu = g_{\mu\nu} w^\nu .$$

Since the application induced by the metric is an isomorphism *we are not going to distinguish any more between vectors and covectors*, i.e. we are going to consider a vector and the associated covector the same geometrical object. This identification is reproduced in the notation by using the same name for the components of a vector and of the associated covector, but with the component index in a different place (respectively *up* and *down*). We are going thus to write in place of the last equation the following one:

$$w_\mu = g_{\mu\nu} w^\nu .$$

In this sense the metric tensor can be used to *lower* the index of a vector (or in general the upper indices of a tensor).

7. Since the metric is non-degenerate, the associated matrix (when we fix basis), which we represent by $g_{\mu\nu}$, has an inverse, i.e. a matrix $(g^{-1})_{\mu\nu}$ such that (without any summation convention)

$$\sum_{\mu}^{0,3} (g^{-1})_{\alpha\mu} g_{\mu\beta} = \sum_{\mu}^{0,3} g_{\alpha\mu} (g^{-1})_{\mu\beta} = \delta_{\alpha\beta} .$$

We are going to write $g^{\mu\nu}$ for the inverse matrix of $g_{\mu\nu}$, i.e. $g^{\mu\nu} \stackrel{\text{def.}}{=} (g^{-1})_{\mu\nu}$ so that the last equation will be written (summation convention now!)

$$g^{\alpha\mu} g_{\mu\beta} = g_{\beta\mu} g^{\mu\alpha} = \delta^\alpha_\beta .$$

[1]Remember that $g_{\mu\nu}$ is symmetric, i.e. $g_{\mu\nu} = g_{\nu\mu}$

In the same way in which $g_{\mu\nu}$ can be used to lower indices, $g^{\mu\nu}$ can be used to *raise* indices. This means that given any tensor, by the natural extension of the isomorphism we defined in the previous point, we are going to consider $T^{\alpha\beta}{}_{\mu\nu}{}^{\gamma}{}_{\rho\sigma}$ and $T_{\lambda}{}^{\beta\xi\tau}{}_{\delta\rho\sigma}$, for example, as the components of the same geometric object (the tensor, indeed). These components are related as

$$T^{\alpha\beta}{}_{\mu\nu}{}^{\gamma}{}_{\rho\sigma} = g^{\alpha\lambda} g_{\xi\mu} g_{\tau\nu} g^{\gamma\delta} T_{\lambda}{}^{\beta\xi\tau}{}_{\delta\rho\sigma},$$

i.e. by raising and lowering indices.

8. A Riemannian manifold has a natural, volume element. In terms of the determinant of the metric, which we will call g, we will write it as $\sqrt{-g} d^4 x$. This volume element is associated to a volume form, $\boldsymbol{\varepsilon}$: this is a totally antisymmetric tensor of rank 4, whose components are $\varepsilon_{\mu\nu\rho\sigma} = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma}$, where $\epsilon_{0123} = +1$ and $\epsilon_{\pi(0)\pi(1)\pi(2)\pi(3)} = \mathrm{sign}\,(\pi)$ where π is a permutation of the set $\{0, 1, 2, 3\}$.

9. As a standard convention vectors, which are $(1, 0)$-tensors, are called **contravariant** vectors since their components transform according to the inverse Jacobean of a coordinate transformation. $(0, 1)$-tensors are instead called **covariant** vectors, since their components transform according to the Jacobean of a coordinate transformation. The metric tensor and its inverse perform the passage between covariant and contravariant components of a tensor object. The same terminology applies to indices in components. Upper indices are called contravariant indices, whereas lower indices are called covariant indices.

16.2 General Relativity

In this section we are going to quickly cover the basic principles underlying the theory of General Relativity. Additional material can be found in many textbooks and we are mainly going to concentrate on the basic, fundamental principles. As we did in the case of Special Relativity, we will mostly quote Einstein directly. We will recognize a similar pattern, in the discussion, as we found in the discussion leading to the special theory. Two problems, i) the generalization of special relativity to arbitrary, not necessarily inertial, reference systems and ii) the description of the gravitational field (impossible in the special theory) will be solved at once and, again, using an experimentally very well established principle, the *equivalence principle*. This principle, as witnessed by Einstein words, is the crucial link between inertial and gravitational effects.

16.2.1 Problems of the special theory

We have seen that the origin of Special Relativity was mainly related to the difficulties that classical mechanics faced at the appearance of Electrodynamics on the scenario of Physics. These difficulties were summarized by Einstein in what he called the *apparent incompatibility* between the principle of special relativity and the law of propagation of light *in vacuum*. Special Relativity solved this *apparent contradiction* by a deep reflection about the concepts of space and time, as we have seen. Although this was already a big step, if compared with the situation present in pre-relativistic physics, the new theory was not free of problems. We are going to discuss this in what follows, although it is our idea that the effort made by Einstein to generalize the theory of Special Relativity were also justified (also by himself) as the effort to complete the theory with the passage from a theory valid in inertial reference system to a theory valid in whatever reference system one could choose. Indeed in pre-relativistic physics, as well as Special Relativity, we have to make a clear cut distinction between inertial reference systems, in which laws of nature are valid, and all other reference system, in which the same laws of nature do not hold.

> But no person whose mode of thought is logical can rest satisfied with this condition of things. [.... There is not] a real something in classical mechanics (or in the special theory of relativity) to which we can attribute the different behavior of bodies considered with respect to [two different reference systems].

Inertial reference systems

The above reflections are more and more reinforced if we think at the first law of motion in Newtonian mechanics. This law, which defines the *free* motion of a non-interacting body as rectilinear and uniform, *at the same time* singles out the class of inertial reference systems: this are defined as the class of systems in which Newton first law holds. We do not want to enter into the details of this problem, just we stress out that the goal of the first law is a too heavy duty for it. From our point of view we want just to stress that this laws makes the concept of free motion and of inertial reference system, conceptually equivalent. This law could have had a strong motivation in a physical realm, as the Newtonian one, where absolute space and time where the preferred arena for dynamics. But with the advent of special Relativity and of the new interpretation of space and time as relative, the weakness of the concept of inertial systems also comes to light.

The principle of general covariance

This is why the requirement of invariance with respect to the class of inertial systems becomes not well funded. The easiest conceptual way to overcome all these problems is to enlarge the class of admissible reference systems. Or, better, to make it as big as possible. This is the content of the
Principle of General Covariance: *all reference systems are equivalent for the description of natural phenomena.*

Gravitation and the weak equivalence principle

To come to an understanding of the relevance of the theory of General Relativity as both a theory that realizes the principle of general covariance and as a theory of gravitational phenomena, we give here a quick remind about a very important property of gravitation. In particular, a remind of what is called
The weak equivalence principle: *the inertial mass of a body equals its gravitational mass.*
This principle needs a reformulation, to make the statement more clear in the context of what will be General Relativity, but we are going to discuss it shortly

here in this form. In particular we want to make more precise what is the inertial mass of a body and what is its gravitational mass. We also remember, referring the reader to the specialized literature on the subject, that the equivalence principle expressed in this form is one of the **experimentally better established** results in physics.

Let us consider a body, which we will label by a "b" in round brackets (b), sometimes as a superscript, $^{(\mathrm{b})}$.
The *Inertial mass*, indicated by $m_{\mathrm{I}}^{(\mathrm{b})}$, is a property that describes how the body (b) behaves when we try to change its velocity. It is the mass that appears in Newton second law, i.e. the constant of proportionality between the force $\vec{F}$ applied to the body (b) and the change in velocity, i.e. the acceleration $\vec{a}$, that (b) acquires because of the force that is acting on it. In an equation this is expressed in the well known Newton second law:

$$\vec{F} = m_{\mathrm{I}}^{(\mathrm{b})}\vec{a}. \tag{16.1}$$

Its *gravitational mass*, indicated by $m_{\mathrm{G}}^{(\mathrm{b})}$, is a property of the body (b) that describes how it responds to a gravitational force. In modern language we could call it the *coupling constant* of the body (b) with the gravitational field. If we consider our body (b) close to the Earth's surface the force acting on it can be expressed in terms of its gravitational mass and of the acceleration $\vec{g}$, which for a motion close enough to the Earth surface is a constant. Thus the gravitational force on our body (b) is

$$\vec{F}_{\mathrm{G}} = m_{\mathrm{G}}^{(\mathrm{b})}\vec{g}. \tag{16.2}$$

Now let us apply Newton second law to our body (b) to find its equation of motion. Let us assume that on (b) only the gravitational force is acting. Then the total force $\vec{F}$ in equation (16.1) is nothing but the gravitational force given by equation (16.2), so that we obtain

$$m_{\mathrm{G}}^{(\mathrm{b})}\vec{g} = m_{\mathrm{I}}^{(\mathrm{b})}\vec{a},$$

or, which is the same,

$$\vec{a} = \frac{m_{\mathrm{G}}^{(\mathrm{b})}}{m_{\mathrm{I}}^{(\mathrm{b})}}\vec{g}. \tag{16.3}$$

Now the equivalence principle states that, *by choosing properly the units of measurement*, we can always have the equality $m_{\mathrm{I}} = m_{\mathrm{G}}$, so that (16.3) simplifies to

$$\vec{a} = \vec{g}. \tag{16.4}$$

This is the equation of motion of our body (b), acted upon by the gravitational field close to Earth surface. What if we choose another body (b′)? Since in (16.4) no reference to the body appears anymore (because of the equivalence principle!) the equation of motion for (b′) will be again (16.4)! The gravitational force treats all body in the same way, whatever their mass (gravitational or inertial is the same, so we do not have to specify which). This crucial property of the gravitational field that gives (16.4) for all bodies, will be used by Einstein to tackle the problem of generalizing special relativity to a general covariant theory **and**, **at the same time**, provide a theory for the gravitational field. We stress again that this possibility has its foundation on both a very well established physical law (the equivalence principle) and on a very strong logical necessity (principle of general covariance). The mix of this two principles is done, in the next section, following (literally) Einstein's teaching in the elevator thought experiment.

16.2.2 Einstein's elevator thought experiment

We imagine a large portion of empty space, so far removed from stars and other appreciable masses, that we have before us approximately the conditions required by the fundamental law of Galilei. [1] It is then possible to choose a Galilean reference-body for this part of space (world), relative to which points at rest remain at rest and points in motion continue permanently in uniform rectilinear motion. As reference-body let us imagine a spacious chest resembling a room with an observer inside who is equipped with apparatus. Gravitation naturally does not exist for this observer. He must fasten himself with strings to the floor, otherwise the slightest impact against the floor will cause him to rise slowly towards the ceiling of the room.

[1] Note from this sentence that Einstein strongly feels the difficulty of defining an inertial reference system, as we tried to quickly discuss above.

To the middle of the lid of the chest is fixed externally a hook with rope attached, and now a "being" (what kind of a being is immaterial to us) begins pulling at this with a constant force. The chest together with the observer then begin to move "upwards" with a uniformly accelerated motion. In course of time their velocity will reach unheard-of values — provided that we are viewing all this from another reference-body which is not being pulled with a rope.

But how does the man in the chest regard the process? [2] The acceleration of the chest will be transmitted to him by the reaction of the floor of the chest. He must therefore take up this pressure by means of his legs if he does not wish to be laid out full length on the floor. He is then standing in the chest in exactly the same way as anyone stands in a room of a home on our earth. If he releases a body which he previously had in his land, the acceleration of the chest will no longer be transmitted to this body, and for this reason the body will approach the floor of the chest with an accelerated relative motion. The observer will further convince himself that *the acceleration of the body towards the floor of the chest is always of the same magnitude, whatever kind of body he may happen to use for the experiment.* [3]

[2] We are making a general coordinate transformation here!

[3] This is a crucial observation, as emphasized in the coming sentence.

Relying on his knowledge of the gravitational field [... as we discussed above], the man in the chest will thus come to the conclusion that he and the chest are in a gravitational field which is constant with regard to time. [4] Of course he will be puzzled for a moment as to why the chest does not fall in this gravitational field. Just then, however, he discovers the hook in the middle of the lid of the chest and the rope which is attached to it, and he consequently comes to the conclusion that the chest is suspended at rest in the gravitational field. [5]

[4] The weak equivalence principle is in effect now! We cannot distinguish between an inertial and a gravitational force!

[5] This is the **crucial point**. The person in the chest has no reason to doubt a gravitational field is acting in the space around him; all experiments he can perform are consistent with this, **thus** his explanation of the situation **is, and must be, ok!** Thus, please, do not smile at him!

Ought we to smile at the man and say that he errs in his conclusion? I do not believe we ought to if we wish to remain consistent; we

> must rather admit that his mode of grasping the situation violates neither reason nor known mechanical laws. Even though it is being accelerated with respect to the "Galilean space" first considered, we can nevertheless regard the chest as being at rest. We have thus good grounds for extending the principle of relativity to include bodies of reference which are accelerated with respect to each other, and as a result we have gained a powerful argument for a generalized postulate of relativity.
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> We must note carefully that the possibility of this mode of interpretation rests on the fundamental property of the gravitational field of giving all bodies the same acceleration, or, what comes to the same thing, on the law of the equality of inertial and gravitational mass. If this natural law did not exist, the man in the accelerated chest would not be able to interpret the behavior of the bodies around him on the supposition of a gravitational field, and he would not be justified on the grounds of experience in supposing his reference-body to be "at rest".

From this short discussion we see that using the equivalence principle we can *trade* a (*uniform*) gravitational field for an inertial field; We can give a unified interpretation to inertial and gravitational phenomena, if we allow general (i.e. also non-inertial) reference system. We can thus have the hope of dealing with the problem of a consistent description of gravitation inside the structure of spacetime by allowing for arbitrary reference systems, i.e. by generalizing special relativity in the direction of general covariance.

16.3 Synopsis and a word of caution

We discussed before special relativity. But some problems still remain open in this new theory. In particular the definition of an inertial reference system is quite elusive, which would make us prefer a description in terms of arbitrary coordinate systems (*general covariance*). At the same time, although the framework of special relativity is quite well adapted to classical electrodynamics, we cannot give in this framework a consistent description of *gravitation*. A possibility of a unifying solution for both problems arises when, on the basis of the *equivalence principle*, we see that in a formulation that admits accelerated systems, we can trade the uniform field of inertial acceleration for a uniform gravitational field.

For the sake of precision we want to stress that although **a uniform gravitational field** *is equivalent with* **a uniform inertial field of acceleration** this equivalence **does not hold** for general gravitational fields. I.e. although we can always choose a reference system in which we do not experience any effect related to a uniform gravitational field (just by choosing to "freely fall" in the gravitational field) this is not true for generic gravitational fields. In the general situation, as we will see, we will still be able to compensate the effects of a gravitational field at a point or, which is empirically more significative, in a sufficient small region of space for a sufficiently short time, i.e. in a sufficiently small region of spacetime. Often instead of "in a sufficiently small region of spacetime" we are going to say *locally*.