Chapter 9

Lecture 9

9.1 Some algebraic preliminaries

In this section we are going to recall the idea of scalar product on a vector space. Although the concept is well known from Algebra courses, we are going to give a slightly different set of definitions. These are motivated by the fact that we will be interested in pseudo-Euclidean scalar product, since they naturally arise in relativity. Pseudo-Euclidean scalar products are non-singular and bilinear but are *not* positive definite.

9.1.1 Scalar products on a vector space

Definition 9.1 (Scalar product)
A real scalar product *over V is a map*

$$\langle -, - \rangle : V \times V \longrightarrow \mathbb{R}$$

which is

1. *symmetric, i.e.* $\forall \boldsymbol{v}, \boldsymbol{w} \in V$ *it satisfies* $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{w}, \boldsymbol{v} \rangle$;
2. *linear in the first argument, i.e.* $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ *and* $\lambda, \mu \in \mathbb{R}$ $\Rightarrow \langle \lambda \boldsymbol{u} + \mu \boldsymbol{v}, \boldsymbol{w} \rangle = \lambda \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \mu \langle \boldsymbol{v}, \boldsymbol{w} \rangle$;
3. *non-degenerate, i.e. such that given* $\boldsymbol{v} \in V$, $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0, \forall \boldsymbol{w} \in V \Rightarrow \boldsymbol{v} = \boldsymbol{0}$.

Given $\{\boldsymbol{e}_i\}_{i=1,\dots,m}$ a basis of V if we consider the matrix $g_{ij} = \langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle$ the symmetry assumption implies $g_{ij} = g_{ji}$ and the non-degenerate assumption implies that the matrix g_{ij} is non singular. A scalar product will be called a *metric* on V. When, given a vector $\boldsymbol{v} = \sum_i^{1,n} v^i \boldsymbol{e}_i$, we consider the map

$$\langle \boldsymbol{v}, - \rangle : V \longrightarrow V$$

this is a linear map on V, i.e. $\langle \boldsymbol{v}, - \rangle \in V^* = \Lambda^1(V)$. We can easily determine its components in the dual basis $\{\boldsymbol{E}^j\}_{j=1,\dots,m}$ by writing

$$\langle \boldsymbol{v}, - \rangle = \sum_j^{1,n} \tilde{v}^j \boldsymbol{E}_j$$

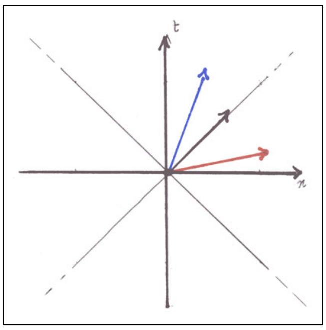

Figure 9.1: Timelike, spacelike and null vectors.

and acting with both sides on $\boldsymbol{w} = \sum_k^{1,n} w^k \boldsymbol{e}_k$:

$$
\begin{aligned}
&= \quad \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \sum_j^{1,n} \tilde{v}_j \boldsymbol{E}^j(\boldsymbol{w}) \quad = \\
\sum_{i,j}^{1,n} g_{ij} v^i w^j &= \qquad\qquad = \sum_j^{1,n} \tilde{v}_j \boldsymbol{E}_j \left(\sum_k^{1,n} w^k \boldsymbol{e}_k \right) \\
\sum_{i,j}^{1,n} g_{ij} v^i w^j &= \qquad\qquad = \sum_{j,k}^{1,n} \tilde{v}_j w^k \boldsymbol{E}_j \left(\boldsymbol{e}_k \right) \\
\sum_j^{1,n} \left(\sum_i^{1,n} g_{ij} v^i \right) w^j &= \qquad\qquad = \sum_j^{1,n} \tilde{v}_j w^j. \qquad (9.1)
\end{aligned}
$$

Thus

$$\tilde{v}_j = \sum_i^{1,n} g_{ij} v^i.$$

The converse is also true: if we have a 1-form $\boldsymbol{\omega} = \sum_i^{1,n} \omega_i \boldsymbol{E}^i \in V^*$ we can associate to it a unique vector $\boldsymbol{w} \in V$, whose components are defined as $w^i = \sum_j^{1,n} (g^{-1})_{ij} \omega_j$. Thus the metric induces a natural isomorphisms between V and V^*. Since the action of an $\boldsymbol{\omega} \in V^*$ is independent from the definition of a metric on V, we will keep the notation $\boldsymbol{\omega}(\boldsymbol{v})$ and we will not rewrite it in terms of the scalar product.

Definition 9.2 (Signature and Lorentzian metric)
Let $\langle -, - \rangle$ be a metric on V. The signature *of the metric is the number of positive eigenvalues of the matrix g_{ij} minus the number of negative eigenvalues. A metric of signature $m - 2$ is called a* Lorentzian metric.

Definition 9.3 (Timelike, spacelike and null vectors)
Let $\langle -, - \rangle$ be a Lorentzian metric on the vector space V. A vector $\boldsymbol{v} \in V$ is timelike *if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0$,* spacelike *if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$ and* null *if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$.*

9.2 Riemannian (Lorentzian) geometry - 1 -

With a procedure that mimics what we already did for tensors, we will now apply the above framework to each tangent space in a given manifold and define a Riemmanian/Lorentzian structure on the manifold itself.

9.2.1 Riemannian and Lorentzian manifolds

Definition 9.4 (Riemannian metric)
Let us consider a manifold $(\mathscr{M}, \mathcal{F})$ *and the set*

$$^{m}\langle \mathscr{M} \rangle = \bigcup_{m \in \mathscr{M}} \{\langle -, - \rangle_{m} \,|\, \langle -, - \rangle_{m} \text{ a positive definite metric on } \mathscr{M}_{m}\}.$$

A differentiable map

$$\langle -, - \rangle : \mathscr{M} \longrightarrow {}^{m}\langle \mathscr{M} \rangle,$$

defined as

$$\langle -, - \rangle (m) \stackrel{\text{def.}}{=} \langle -, - \rangle_{m},$$

is called a Riemannian metric *on* $\mathscr{M}$*.*

Differentiability is defined, as usual, in terms of vector fields, i.e. $\langle -, - \rangle$ is differentiable if for every choice of vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ on an open set $U \subset \mathscr{M}$ the function

$$\langle \boldsymbol{V}, \boldsymbol{W} \rangle : U \longrightarrow \mathbb{R},$$

defined as $\langle \boldsymbol{V}, \boldsymbol{W} \rangle (\mathtt{m}) \stackrel{\text{def.}}{=} \langle \boldsymbol{V}_{\mathtt{m}}, \boldsymbol{W}_{\mathtt{m}} \rangle_{\mathtt{m}}$, is differentiable.

Proposition 9.1 (Existence of Riemannian metric)
Every differentiable manifold admits a Riemannian metric.

Proof:

The proof of this statement proceeds along the same line we used for the characterization of the orientation on a manifold. Let thus $(\mathscr{M}, \mathcal{F})$ be a differentiable manifold, of dimension m. Let us consider a partition of unity $(\mathscr{R}, \mathcal{P})$ subordinated to the cover $\mathcal{U} = \{U |\ (U, \phi) \in \mathcal{F}\}$. For each $\mathtt{m} \in \mathscr{M}$, $\exists V \in \mathscr{R}$ such that $\mathtt{m} \in V$. Moreover $\exists U \in \mathcal{U}$ such that $V \subset U$ so that $\forall \mathtt{m} \in V \subset U$ in $\mathscr{M}_{\mathtt{m}}$ the coordinate map ϕ associated to U with coordinate functions $x_1, \ldots, x_m$ induces the coordinate basis $\{\partial / \partial x_i]_{\mathtt{m}} \}_{i=1,\ldots,m}$. Thus $\forall \mathtt{m} \in V$ we can define a scalar product $\langle -, - \rangle_V$ by

$$\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_V = \delta_{ij}.$$

Then

$$\langle -, - \rangle \stackrel{\text{def.}}{=} \sum_{V \in \mathscr{R}} f_V \langle -, - \rangle_V$$

is a Riemannian metric on $\mathscr{M}$.

□

Definition 9.5 (Lorentzian metric)
Let us consider a manifold $(\mathscr{M}, \mathcal{F})$ *and the set*

$$^{m-2}\langle \mathscr{M} \rangle = \bigcup_{m \in \mathscr{M}} \{ \langle -, - \rangle_m \,|\, \langle -, - \rangle_m \, a \text{ metric of signature } m-2 \text{ on } \mathscr{M}_m \} .$$

A differentiable map

$$\langle -, - \rangle : \mathscr{M} \longrightarrow {}^{m-2}\langle \mathscr{M} \rangle,$$

defined as

$$\langle -, - \rangle (m) \stackrel{\text{def.}}{=} \langle -, - \rangle_m ,$$

is called a Lorentzian metric *on* $\mathscr{M}$.

In what follows if we will refer to a metric, without specifying if it is Riemannian or Lorentzian, we will assume that the type of metric is not relevant, e.g. the results hold for the Riemannian as well as for the Lorentzian case.

Proposition 9.2 (Existence of Lorentzian metric)
A paracompact manifold admits a Lorentzian metric if and only if it admits a non-vanishing line element field.

Definition 9.6 (Isometry between manifolds)
Let $\mathscr{M}, \mathcal{F}$ *and* $\mathscr{N}, \mathcal{G}$ *be two differentiable manifold and* $\phi : \mathscr{M} \longrightarrow \mathscr{N}$ *a map between them.* ϕ *is an* isometry *if it is a diffeomorphism and if its differential* $d\phi$ *is a vector space isometry* $\forall m \in \mathscr{M}$, *i.e. if* $\forall m \in \mathscr{M}$

$$\langle d\phi(\boldsymbol{v}), d\phi(\boldsymbol{w}) \rangle_{\phi(m)} = \langle \boldsymbol{v}, \boldsymbol{w} \rangle_m \quad , \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathscr{M}_m.$$

9.3 Connections on manifolds - 1 -

In the past lectures we have seen some concepts connected to the ideas of field in the sense of mathematical physics. Fields for us will be tensor fields on manifolds. Remembering what we said at the very beginning our fields, are the variables in term of which we define a theory. From what we already know from the courses of physics, we understand that knowing the variables of a problem is not enough. We have to define concepts that can be associated to measurable quantities and write equations for them (equations like Newton equation, for example). From our experience we know that these equations, are in very broad terms, differential equations, i.e. equations relating the value of the fields and of their derivatives. We have the fields, but how about the derivatives? We still lack this concept on a manifold. To define it in a proper way is our next goal. How can we possibly do that? To get a clue we think of what we do in the Euclidean space $\mathbb{R}^3$. Let us consider a vector field in $\mathbb{R}^3$, for example the velocity field along the line described by a moving particle. How can we define the acceleration starting from it? The acceleration is again a vector field. We can obtain it by taking the velocity of the particle $\boldsymbol{v}(t)$ at some instant t. Then the velocity of the particle $\boldsymbol{v}(t + \Delta t)$ at some later instant $t + \Delta t$. Then we compute the difference of them, we divide by Δt and we take the limit of $\Delta t \to 0$. We obtain

$$\boldsymbol{a}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{v}(t + \Delta t) - \boldsymbol{v}(t)}{\Delta t},$$

i.e. the acceleration. Let us think carefully about what we did. There is a difference at the numerator of the above expression, between two vectors:

1. $\boldsymbol{v}(t+\Delta t)$, a vector at the point $P(t+\Delta t)$, which is the point where the particle is at the instant $t+\Delta t$;
2. $\boldsymbol{v}(t)$, a vector at the point $P(t)$, which is the point where the particle is at the instant t.

We make the difference between two vectors at two **different points**? This cannot be, since the difference is **not** an operation defined between vectors in **two different** vector spaces. If V and W are two different vector spaces and $\boldsymbol{v} \in V$, $\boldsymbol{w} \in W$ there is no vector space operation like $\boldsymbol{v}-\boldsymbol{w}$. So what is the real meaning of the minus sign in the expression $\boldsymbol{v}(t+\Delta t) - \boldsymbol{v}(t)$? Well, actually there is not *one* meaning, there are *two*:

1. we have to move the vector $\boldsymbol{v}(t+\Delta t)$ from the point $P(t+\Delta t)$ to the point $P(t)$, i.e. move the vector $\boldsymbol{v}(t+\Delta t)$ from the vector space $(\mathbb{R}^3)_{P(t+\Delta t)}$ of all vectors defined at $P(t+\Delta t)$ to the vector space $(\mathbb{R}^3)_{P(t)}$ of all vectors defined at $P(t)$;
2. we then subtract from **the transferred vector** the vector $\boldsymbol{v}(t)$, since now they are in the same vector space.

Another question now arises. How do we have to transfer the vector $\boldsymbol{v}(t+\Delta t)$ from $(\mathbb{R}^3)_{P(t+\Delta t)}$ to $(\mathbb{R}^3)_{P(t)}$? Well, we can just keep it *parallel to itself*, you could answer, so that we have exactly the same vector at the new point. But then what is the meaning of *parallel*? Maybe we implicitly give a meaning to this expression in $\mathbb{R}^3$. But how we can generalize this concept on a manifold for our tensor fields? Yes, **we need a way to connect tensors at one point to tensors at another point, to be able to appreciate the difference between them and define derivatives**! On a generic manifold we do not have the **global** power of the Euclidean structure of $\mathbb{R}^3$. We have to find another way (or many other ways) to meaningfully define how to *parallel translate* a tensor from the vector space of tensors at one point, to the vector space of tensors at another point. We need a concept that *connects* spaces of tensors at different points; moreover, to be a good generalization of what we are already doing in $\mathbb{R}3$, it must indeed give the usual result in $\mathbb{R}^3$. This concept, which we are going to study thoroughly, will be a **connection**. We will define the concept for vector fields first and we will generalize it to tensor fields later on.

9.3.1 Connections and symmetric connections

Definition 9.7 (Connection at $\mathtt{m} \in \mathscr{M}$)
Let $\mathscr{M}$ be a differentiable manifold. A connection at $m \in \mathscr{M}$ *is a map*

$$D(-,-)_m : \mathscr{M}_m \times \mathcal{V}(\mathscr{M}) \longrightarrow \mathscr{M},$$

such that:

1. *$D(\boldsymbol{v}_m, \boldsymbol{W})_m$ is bilinear in $\boldsymbol{v}_m$ and $\boldsymbol{W}$;*
2. *$\forall f : \mathscr{M} \longrightarrow \mathbb{R}$ differentiable,*

$$D(\boldsymbol{v}_m, f\boldsymbol{W})_m = \boldsymbol{v}_m(f)\boldsymbol{W}_m + f(m) D(\boldsymbol{v}_m, \boldsymbol{W})_m.$$

$D(\boldsymbol{v}_{\mathtt{m}}, \boldsymbol{W})_{\mathtt{m}}$ is called the *covariant derivative* of the vector field W in the direction of $\boldsymbol{v}_{\mathtt{m}}$ at $\mathtt{m}$.

Definition 9.8 (Connection on a manifold)
Let $\mathscr{M}$ be a differentiable manifold. A connection on $\mathscr{M}$ *is a map*

$$D(-,-) : \mathcal{V}(\mathscr{M}) \times \mathcal{V}(\mathscr{M}) \longrightarrow \mathcal{V}(\mathscr{M}),$$

such that:

1. $D(\boldsymbol{V}, \boldsymbol{W})$ *is bilinear in* V *and* W*;*
2. $\forall f : \mathscr{M} \longrightarrow \mathbb{R}$ *differentiable,*
$$D(f\boldsymbol{V}, \boldsymbol{W}) = fD(\boldsymbol{V}, \boldsymbol{W});$$
3. $\forall f : \mathscr{M} \longrightarrow \mathbb{R}$ *differentiable,*
$$D(\boldsymbol{V}, f\boldsymbol{W}) = \boldsymbol{V}(f)\boldsymbol{W} + fD(\boldsymbol{V}, \boldsymbol{W}).$$

We have that $\forall \mathtt{m} \in \mathscr{M}$

$$(D(\boldsymbol{V}, \boldsymbol{W}))_{\mathtt{m}} = D(\boldsymbol{V}, \boldsymbol{W})(\mathtt{m}) \stackrel{\text{def.}}{=} D(\boldsymbol{V}_{\mathtt{m}}, \boldsymbol{W})_{\mathtt{m}}$$

where $D(\boldsymbol{V}_{\mathtt{m}}, \boldsymbol{W})_{\mathtt{m}}$ is a connection at $\mathtt{m} \in \mathscr{M}$.

Definition 9.9 (Symmetric connection)
Let $\mathscr{M}, \mathcal{F}$ be a manifold and $D(-,-)$ a connection on $\mathscr{M}$. D is symmetric *if $\forall \boldsymbol{V}$, $\boldsymbol{W}$ vector fields on $\mathscr{M}$, then*

$$D(\boldsymbol{V}, \boldsymbol{W}) - D(\boldsymbol{W}, \boldsymbol{V}) = [\boldsymbol{V}, \boldsymbol{W}].$$

Definition 9.10 (Connection in coordinates)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold of dimension $\dim(\mathscr{M}) = m$ *with connection $D(-,-)$ and let $(U, \phi) \in \mathcal{F}$ with coordinate functions $x^1, \ldots, x^m$. Then in the chart (U, ϕ) we have*

$$D\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_k^{1.m} \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

with

$$\Gamma^k_{ij} : U \longrightarrow \mathbb{R}$$

differentiable functions on $U \subset \mathscr{M}$.