Chapter 2

Lecture 2

2.1 From discrete to continuous systems

Let us consider the following discrete system: $N + 1$ particles in 1 space dimension, connected by N springs. Let all the particle masses be identical and equal to m, all the spring constants also identical and equal to k and all the spring rest lengths also identical and equal to Δl. Let $q_i(t)$ be the position of the i-th particle of the system (we choose the $\{q_i(t)\}_{i=1,\dots,N+1}$ as the canonical coordinates of the system). Let us also define another specific set of generalized coordinates, which we will call $\{\theta_i(t)\}_{i=1,\dots,N+1}$ in such a way that:

$$q_j(t) = j\Delta l + \theta_j(t) \quad , \quad j = 1, \dots, N + 1. \tag{2.1}$$

Since we assume that the rest length of all the springs is Δl, note that if we number particles in such a way that they are in the order of the index of the corresponding generalized coordinates when the springs are all at rest, then the $\{\theta_i(t)\}_{i=1,\dots,N+1}$ represent the *displacements of the particles* from the position they would occupy when all springs have their rest length.

In terms of the first set of canonical coordinates the kinetic energy of the system is

$$T_q = \frac{m}{2} \sum_i^{1,N+1} \dot{q}_i(t)^2.$$

In the same coordinates the potential energy turns out to be

$$V_q = \frac{k}{2} \sum_i^{1,N} \left(q_{i+1}(t) - q_i(t) - \Delta l\right)^2,$$

where k is a constant that can be expressed as

$$k = \frac{\kappa}{\Delta l} \tag{2.2}$$

in terms of a constant κ, with the units of a force, and Δl, the rest length of the spring. The Lagrangian of the system is thus

$$L_q = T_q - V_q = \frac{1}{2}\left[m \sum_i^{1,N+1} \dot{q}_i(t)^2 - k \sum_i^{1,N} \left(q_{i+1}(t) - q_i(t) - \Delta l\right)^2\right].$$

The corresponding conjugate momenta are the derivatives of the Lagrangian with respect to the generalized velocities, i.e.

$$p_j(t) = \frac{\partial L_q}{\partial \dot{q}_j(t)} = m\dot{q}_j(t).$$

We now turn to the equations of motion of the system. The potential is the only part of the Lagrangian depending on the generalized coordinates, so that

$$\begin{aligned}
\frac{\partial L_q}{\partial q_j(t)} &= -\partial_{q_j(t)} V_q \\
&= -\frac{k}{2}\partial_{q_j(t)}\left[\sum_i^{1,N}\left(q_{i+1}(t)-q_i(t)-\Delta l\right)^2\right] \\
&= -\frac{k}{2}\left[\sum_i^{1,N}\partial_{q_j(t)}\left(q_{i+1}(t)-q_i(t)-\Delta l\right)^2\right] \\
&= -\frac{k}{2}\left[\sum_i^{1,N}2\left(q_{i+1}(t)-q_i(t)-\Delta l\right)\partial_{q_j(t)}\left(q_{i+1}(t)-q_i(t)-\Delta l\right)\right] \\
&= -\frac{k}{2}\left[\sum_i^{1,N}2\left(q_{i+1}(t)-q_i(t)-\Delta l\right)\left(\delta_{j,i+1}-\delta_{j,i}\right)\right] \\
&= -k\left[\left(q_j(t)-q_{j-1}(t)-\Delta l\right)-\left(q_{j+1}(t)-q_j(t)-\Delta l\right)\right] \\
&= k\left[q_{j+1}(t)-2q_j(t)+q_{j-1}(t)\right].
\end{aligned} \tag{2.3}$$

Thus the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L_q}{\partial \dot{q}_j(t)}\right) - \frac{\partial L_q}{\partial q_j(t)} = 0 \quad , \quad j = 1,\ldots,N+1$$

turn out to be

$$m\ddot{q}_j(t) = k\left[q_{j+1}(t) - 2q_j(t) + q_{j-1}(t)\right] \quad , \quad j = 1,\ldots,N+1.$$

A last quantity we are interested in is the Hamiltonian of the system. Legendre transforming the Lagrangian description we obtain

$$\begin{aligned}
H_q &= \sum_i^{1,N+1}\dot{q}_i(t)p_i(t) - L \\
&= \frac{1}{2m}\sum_i^{1,N+1}p_i(t)^2 + \frac{k}{2}\sum_i^{1,N}\left(q_{i+1}(t)-q_i(t)-\Delta l\right)^2.
\end{aligned}$$

We will now rewrite the same quantities in terms of the canonical variables $\{\theta_i(t)\}_{i=1,\ldots,N+1}$. From the definition (2.1) we see that

$$\dot{q}_j(t) = d_t\left(j\Delta l + \theta_j(t)\right) = \dot{\theta}_j(t)$$

and that

$$\begin{aligned}
q_{j+1}(t) - q_j(t) - \Delta l &= (j+1)\Delta l + \theta_{j+1}(t) - j\Delta l - \theta_j(t) - \Delta l \\
&= \theta_{j+1}(t) - \theta_j(t).
\end{aligned}$$

Thus no dramatic changes occur in the expressions for the kinetic energy, potential energy and Lagrangian, which turn out to be[1]

$$
\begin{aligned}
T_\theta &= \frac{m}{2}\sum_i^{1,N+1}\dot{\theta}_i(t)^2 \\
V_\theta &= \frac{k}{2}\sum_i^{1,N}\left(\theta_{i+1}(t)-\theta_i(t)\right)^2 \\
L_\theta &= T_\theta - V_\theta = \frac{1}{2}\left[m\sum_i^{1,N+1}\dot{\theta}_i(t)^2 - k\sum_i^{1,N}\left(\theta_{i+1}(t)-\theta_i(t)\right)^2\right]. \qquad (2.4)
\end{aligned}
$$

Accordingly we can calculate the conjugate momenta as

$$\pi_j(t) = \frac{\partial L_\theta}{\partial \dot{\theta}_j(t)} = m\dot{\theta}_j(t) \qquad (2.5)$$

and the Euler-Lagrange equations as

$$m\ddot{\theta}_j(t) = k\left(\theta_{j+1}(t) - 2\theta_j(t) + \theta_{j-1}(t)\right) \quad , \quad j = 1, \ldots, N+1. \qquad (2.6)$$

Finally the Hamiltonian is

$$
\begin{aligned}
H_\theta &= \sum_i^{1,N+1}\dot{\theta}_i(t)\pi_i(t) - L \\
&= \frac{1}{2m}\sum_i^{1,N+1}\pi_i(t)^2 + \frac{k}{2}\sum_i^{1,N}\left(\theta_{i+1}(t)-\theta_i(t)\right)^2. \qquad (2.7)
\end{aligned}
$$

We carried on the computations until this point in the two different generalized coordinate systems: the notation of the first one is more usual but for the interpretation we are going to develop in what follows the second one is more meaningful. As we have shown all the relevant quantities maintain the same form, since we are simply doing a specific constant translation of the origin for each particle's coordinate. As we anticipated above, our coming interpretation requires the second set of generalized coordinates, so from now on we are going to stich with the description of the system in term of the *displacements* $\{\theta_i(t)\}_{i=1,\ldots,N+1}$. In particular we want to consider the situation in which **keeping the *total length* and *total mass* of the system *fixed*, we increase the number of masses (and, consequently, of springs)**: $N \to \infty$. Let

$$M = (N+1)m \quad \text{and} \quad L = N\Delta l$$

be the fixed total mass and total length of the system. We can rewrite the Lagrangian (2.4) as

$$L_\theta^{(N,\Delta l)} = \frac{1}{2}\left[\frac{m(N+1)}{\Delta l(N+1)}\sum_i^{1,N+1}\dot{\theta}_i(t)^2\Delta l - k\Delta l\sum_i^{1,N}\left(\frac{\theta_{i+1}(t)-\theta_i(t)}{\Delta l}\right)^2\Delta l\right]$$

[1]As we used a subscript q to identify some quantities written in terms of the $\{q_i(t)\}_{i=1,\ldots,n}$ coordinate system, we are going to use a subscript θ to identify quantities written in terms of the $\{\theta_i(t)_{i=1,\ldots n}$ coordinate system.

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{M}{L+\Delta l}\sum_{i}^{1,N+1}\dot{\theta}_i(t)^2\Delta l - k\Delta l\sum_{i}^{1,N}\left(\frac{\theta_{i+1}(t)-\theta_i(t)}{\Delta l}\right)^2\Delta l\right] \\
&= \frac{M}{2(L+\Delta l)}\sum_{i}^{1,N+1}\dot{\theta}_i(t)^2\Delta l - \frac{\kappa}{2}\sum_{i}^{1,N}\left(\frac{\theta_{i+1}(t)-\theta_i(t)}{\Delta l}\right)^2\Delta l, \qquad (2.8)
\end{aligned}
$$

where we emphasize that it explicitly depends on the number of masses, N, and on the spring length Δl. Let us now define a function $\Theta(l;t)$ in such a way that for $j = 1, \ldots, N$ the following relations are satisfied:

$$
\theta_j(t) \stackrel{\text{def.}}{=} \Theta(j\Delta l; t) \quad \text{which we will identify with} \quad \Theta(l;t). \qquad (2.9)
$$

Then we also have for $j = 1, \ldots, N$:

$$
d_t\theta_j(t) = \partial_t\Theta(j\Delta l; t) \quad \text{which we will identify with} \quad \partial_t\Theta(l;t).
$$

According to the above relations and identifications the Lagrangian (2.8) becomes

$$
L_\theta^{(N,\Delta l)} = \frac{M}{2(L+\Delta l)}\underbrace{\sum_{i}^{1,N+1}\left(\partial_t\Theta(l;t)\right)^2\Delta l}_{} - \frac{\kappa}{2}\underbrace{\sum_{i}^{1,N}\left(\frac{\Theta(l+\Delta l;t)-\Theta(l;t)}{\Delta l}\right)^2\Delta l}_{}\,.
$$

We now *heuristically* take the limit $N \to \infty$, $\Delta l \to 0$ by keeping, as we said above, M and L fixed. In this limit[2]

$$
\frac{M}{L+\Delta l} \to \frac{M}{L} = \frac{m}{\Delta l} \stackrel{\text{def.}}{=} \mu \qquad (2.10)
$$

and we define[3]

$$
L(\Theta, \partial\Theta) \stackrel{\text{def.}}{\rightsquigarrow} \lim_{\substack{N\to\infty \\ \Delta l\to 0}} L_\theta^{(N,\Delta l)},
$$

which turns out to be

$$
L(\Theta, \partial\Theta) = \frac{\mu}{2}\underbrace{\int_0^L \left(\partial_t\Theta(l;t)\right)^2 dl}_{} - \frac{\kappa}{2}\underbrace{\int_0^L \left(\partial_l\Theta(l;t)\right)^2 dl}_{}\,. \qquad (2.11)
$$

After performing the above (not always well defined) computations, we have to look back to see which kind of meaning we can associate to the expression (2.11). We can consider, one by one, all the quantities that appear in it. As a preliminary observation we note that in the limit we have considered we have passed from a discrete system (containing a finite number of particles indexed by an index $i \in \mathbb{N}$) to a *continuous* system (containing an infinity of particles indexed by an index $l \in \mathbb{R}$). Then let us analyze in turn the μ and κ parameters and the Θ function.

[2] We are going to use an arrow, "$\to$" to identify quantities for which the considered limit procedure is mathematically rigorously defined. When it is not, so that we just have a correspondence using heuristic arguments, we are going to put this in clear evidence by using the symbol "$\rightsquigarrow$".

[3] We comprehensively denote as $\partial\Theta$ a generic (or the set of all) partial derivative(s) of the function Θ.

The parameter μ: the meaning of the parameter μ can be read off from equation (2.10); it is the total mass of the system, which is kept constant in the considered limit, divided by the total length of the system, which we also kept constant in the limit process. It is thus the *density of mass* of the continuous system.

The parameter κ: our procedure does not affect this parameter which has the dimensions of a force.

The function $\Theta(l;t)$: the function $\Theta(l;t)$ is defined in such a way that the functions of t singled out as the values $\Theta(j\Delta l;t)$ agree with the functions $\theta_i(t)$, that describe the displacement of the i-th particle from the position $i\Delta l$ on the "l-axis". We want to observe that the l variable is just an index, although continuous, which corresponds to the index i, in the discrete case. It is not a *dynamical variable* of the system. The functions $\Theta(l;t)$, one for each value of l, are the dynamical variables of the system. This is witnessed by the fact that the Lagrangian becomes a function**al** of these functions: it will be used to describe the dynamics of the system represented by the continuous set of coordinates $\Theta(l;t)$ with $l \in [0, L] \subset \mathbb{R}$.

2.2 Some naming conventions

In many books and in the literature the following naming conventions apply:

Θ , i.e. the set of variables, is called a *field* (fields if there are more than one);

$L(\Theta, \partial\Theta)$, is the *Lagrangian* of the system; it is a functional, since

$$L(\Theta, \partial\Theta) = \int_0^L \mathcal{L}(\Theta, \partial\Theta) dl.$$

Another important functional is the action functional

$$\int dt L(\Theta, \partial\Theta) = \int \mathcal{L}(\Theta, \partial\Theta) dl dt.$$

$\mathcal{L}(\Theta, \partial\Theta)$, defined above, is called the *Lagrangian density* of the system.

As we have seen from our heuristic derivation, the role of l and t in the field is different (this is because we are working in a non-relativistic theory) and this is the reason why we use a semicolon to separate them, when we explicitly write the dependence of Θ. As we *will* see, in relativistic theories space and time have not a distinguished character and no such distinction will appear in our way of writing the field dependence, which usually looks like $\phi(x)$, $\Psi(x)$, $g_{\mu\nu}(x)$, etc., where x represents a set of **spacetime** coordinates.

2.3 The equations of motion

We are now interested in understanding what equations are obeyed by the field $\Theta(l, t)$. We will derive these equations now by the application of the heuristic limit procedure to the equations of motion (2.6) for the discrete system. In a forthcoming lecture we are going to obtain them as a the *Euler-Lagrange* equations directly from the Lagrangian density of the field.

2.3.1 Field equations

Let us consider the equations of motion (2.6) and the definition (2.9) of the field $\Theta(l;t)$. Starting from this definition, the terms in the right-hand side of (2.6) can be expressed as

$$\begin{aligned} \theta_{j-1}(t) &= \Theta(j\Delta l - \Delta l; t) \\ \theta_j(t) &= \Theta(j\Delta l; t) \\ \theta_{j+1}(t) &= \Theta(j\Delta l + \Delta l; t) \end{aligned}$$

The terms in the middle line does not look awkward, but the first and last lines need a careful treatment. We are going to expand them in Taylor series:

$$\begin{aligned} \theta_{j-1}(t) &= \Theta(j\Delta l - \Delta l; t) \\ &\rightsquigarrow \Theta(l;t) - \partial_l \Theta(l;t)\Delta l + \frac{1}{2}\partial_l^2 \Theta(l;t)(\Delta l)^2 + \mathcal{O}((\Delta l)^3) \\ -2\theta_j(t) &\rightsquigarrow -2\Theta(l;t) \\ \theta_{j+1}(t) &= \Theta(l + \Delta l; t) \\ &\rightsquigarrow \Theta(l) + \partial_l \Theta(l;t)\Delta l + \frac{1}{2}\partial_l^2 \Theta(l;t)(\Delta l)^2 + \mathcal{O}((\Delta l)^3). \end{aligned}$$

Summing the above contributions we get

$$\theta_{j+1}(t) - 2\theta_j(t) + \theta_{j-1}(t) \rightsquigarrow \partial_l^2 \Theta(l;t)(\Delta l)^2 + \mathcal{O}((\Delta l)^3).$$

This deals with the right-hand side of (2.6). For the left-hand side we have more simply

$$\ddot{\theta}_j(t) \rightsquigarrow \partial_t^2 \Theta(l;t),$$

so that, when taking into account (2.10) and (2.2) the equations of motion become

$$\mu \partial_t^2 \Theta(l;t) - \kappa \partial_l^2 \Theta(l;t) = 0. \tag{2.12}$$

All the $N+1$ equations of motion of the system, when $N \to \infty$, become a single equation (by the way, the wave equation in this case). There is, of course, an evident difference when we pay a closer attention at the *kind* of equation(s) that we obtain in the two cases: the $N+1$ equations for the discrete system are ordinary differential equations of the second order in the *unknown* functions $\{\theta_i(t)\}_{i=1,\ldots,N+1}$. To solve them we have to specify, together with the equations, $2(N+1)$ initial conditions, $(N+1)$ for the initial positions $\theta_i^{(0)} = \theta_i(t_0)$ of the masses at time $t = t_0$, say, and $(N+1)$ for the initial velocities $\omega_i^{(0)} = \dot{\theta}_i(t_0)$. The field equation (2.12) is instead a partial differential equation for the *unknown* function $\Theta(l;t)$ of the two variables l and t (where we stress again that l is just a continuous label for the degrees of freedom of the system): it is of the second order in l and of the second order in t. What are the initial conditions? The initial conditions can be inferred with the same *heuristic* procedure above and according to the correspondence

$$\begin{aligned} \theta_i^{(0)} &\rightsquigarrow \Theta^{(0)}(l) \\ \omega_i^{(0)} &\rightsquigarrow \Omega^{(0)}(l) \end{aligned}$$

and where the relations

$$\begin{aligned}\theta_j^{(0)} &= \Theta^{(0)}(j\Delta l)\\ \omega_j^{(0)} &= \Omega^{(0)}(j\Delta l)\end{aligned}$$

hold. They are thus two *functions*, that describe the initial configuration $\Theta(l;t_0)$ and its time derivative $\partial_t\Theta(l;t_0)$ at the instant $t=t_0$:

$$\begin{aligned}\Theta(l;t_0) &= \Theta^{(0)}(l)\\ \partial_t\Theta(l;t_0) &= \Omega^{(0)}(l).\end{aligned}$$

This ends our discussion of the equations of motion for now. More about this can be found in an exercise at the end of the next lecture.

Now we will define some additional dynamical quantities in the continuum limit.

2.4 Other dynamical quantities in field theory

We would like to generalize the concept of momentum and Hamiltonian to the case of the Θ field. The generalization of the momentum (2.5) can be derived applying the above defined heuristic correspondence to the quantity

$$\frac{1}{\Delta l}\frac{\partial L_\theta^{(N,\Delta l)}}{\partial\dot\theta_j}$$

where $L_\theta^{(N,\Delta l)}$ is defined by (2.8). This can be seen by first computing

$$\begin{aligned}\frac{\partial L_\theta^{(N,\Delta l)}}{\partial\dot\theta_j} &= \frac{M}{2(L+\Delta l)}\sum_i^{1,N+1}\frac{\partial\dot\theta_i(t)^2}{\partial\dot\theta_j(t)}\Delta l\\ &= \frac{M}{(L+\Delta l)}\sum_i^{1,N+1}\dot{\theta_i}(t)\delta_{i,j}\Delta l\\ &= \frac{M}{(L+\Delta l)}\dot\Theta(j\Delta l;t)\Delta l\\ &\rightsquigarrow \ldots ?. \end{aligned}\tag{2.13}$$

We could be tempted to substitutes the ellipsis and the question mark in the line above to define the momentum of the field corresponding to $\pi_j(t)$ using the already employed heuristic limit procedure. Unfortunately this apparently *natural* definition gives *zero* since a quantity finite in the limit is multiplied by Δl, which goes to zero. We thus remove the unpleasant Δl factor observing that a finite result is obtained with the quantity we anticipated at the beginning of this section, i.e. by the correspondence

$$\frac{1}{\Delta l}\frac{\partial L_\theta^{(N,\Delta l)}}{\partial\dot\theta_j}\rightsquigarrow\mu\partial_t\Theta(l;t)\tag{2.14}$$

so that we will define the *momentum density*[4] as

$$\Pi(l;t)=\mu\partial_t\Theta(l;t).\tag{2.15}$$

[4]The word density, intuitively, is related to the fact that we divide by Δl.

It satisfies the correspondence

$$\frac{\pi_j(t)}{\Delta l} \rightsquigarrow \Pi(j\Delta l; t).$$

We also observe that starting from the Lagrangian density

$$\mathcal{L}(\Theta, \partial\Theta) = \frac{\mu}{2}\left(\partial_t\Theta(l;t)\right)^2 - \frac{\kappa}{2}\left(\partial_x\Theta(l;t)\right)^2$$

we can write

$$\Pi(l;t) = \frac{\partial\mathcal{L}}{\partial(\partial_t\Theta(l;t))} \tag{2.16}$$

which is the equation corresponding to the first equality in (2.5). It is then plausible that the *Hamiltonian density* can be obtained just as

$$\mathcal{H}(\Theta;\Pi) = \Pi(l;t)\partial_t\Theta(l;t) - \mathcal{L}(\Theta;\partial\Theta), \tag{2.17}$$

where in place of $\partial_t\Theta(l;t)$ we use its expression in terms of $\Pi(l;t)$ that can be obtained inverting (2.15) (in our particular example) or (2.16) (in a more general case). The Hamiltonian density above can also be obtained, using the heuristic limit procedure, from equation (2.7) as follows

$$\begin{aligned}
H_\theta &= \sum_i^{1,N+1} \dot{\theta}_i(t)\pi_i(t) - L_\theta \\
&= \sum_i^{1,N+1} \dot{\theta}_i(t)\frac{\pi_i(t)}{\Delta l}\Delta l - L_\theta \\
&= \sum_i^{1,N+1} \partial_t\Theta(i\Delta l;t)\Pi(i\Delta l;t)\Delta l - L_\theta \\
&\rightsquigarrow \int_0^L \partial_t\Theta(l;t)\Pi(l;t)dl - \int_0^L \mathcal{L}(\Theta,\partial\Theta)dl \\
&\rightsquigarrow \int_0^L \left[\Pi(l;t)\partial_t\Theta(l;t) - \mathcal{L}(\Theta,\partial\Theta)\right]dl;
\end{aligned}$$

the quantity in square brackets is then nothing but (2.17).

2.5 Synopsis

In this lecture we have seen an *heuristic procedure* to go from the Lagrangian or Hamiltonian description of a discrete system to the Lagrangian or Hamiltonian description of an associated continuous system. Although the limit procedure employed is not rigorous, it helps understanding that the continuous variables play the role of the discrete indices, i.e. they index a continuous set of variables. We have also seen that quantities analogous to the Hamiltonian and the Lagrangian are *functionals* of the fields. They are the integrals on the space of parameters of the corresponding densities.