

# Fisica Matematica

Stefano Ansoldi

Dipartimento di Matematica e Informatica

Università degli Studi di Udine

Corso di Laurea in Matematica

Anno Accademico 2003/2004



©2004

Copyright by Stefano Ansoldi and the University of Udine, 2004.

All Italian and international copyrights reserved  
for all original material presented in this course and lecture notes  
through any medium, including lecture or print.

Individuals are prohibited from being paid for taking, selling,  
or otherwise transferring for value, this material or part of it  
without the express written permission of Stefano Ansoldi.

Individuals are prohibited from distributing by any means  
this material or part of it

without the express written permission of Stefano Ansoldi.

Individuals are granted the right of using this material  
for personal, educational purposes only  
without the express written permission of Stefano Ansoldi.



# Contents

<b>Contents</b>	<b>i</b>
<b>List of Figures</b>	<b>v</b>
<b>List of Tables</b>	<b>vii</b>
<b>List of Definitions</b>	<b>ix</b>
<b>List of Propositions</b>	<b>xi</b>
<b>List of Notations</b>	<b>xiii</b>
<b>II Acknowledgments</b>	<b>xv</b>
<b>1 Lecture 1</b>	<b>1</b>
1.1 Introduction . . . . .	1
1.2 A “glimpse” at Mathematical Physics . . . . .	1
1.2.1 Operative definition of physical concepts . . . . .	1
1.2.2 Axiomatic definition of Mathematical Concepts . . . . .	2
1.2.3 Some contact points . . . . .	2
1.3 Our program in Mathematical Physics . . . . .	3
<b>2 Lecture 2</b>	<b>5</b>
2.1 From discrete to continuous systems . . . . .	5
2.2 Some naming conventions . . . . .	9
2.3 The equations of motion . . . . .	9
2.3.1 Field equations . . . . .	10
2.4 Other dynamical quantities in field theory . . . . .	11
2.5 Synopsis . . . . .	12
<b>3 Lecture 3</b>	<b>13</b>
3.1 Euler-Lagrange equations in field theory . . . . .	13
3.1.1 Preliminaries . . . . .	13
3.1.2 Functional Derivatives . . . . .	14
3.1.3 Extremals and field equations . . . . .	14
3.2 Stationary action principle . . . . .	17
3.3 Synopsis . . . . .	18

---

<b>4 Lecture 4</b>	<b>19</b>
4.1 Reflections on space and time . . . . .	19
4.1.1 Classical mechanics and its framework . . . . .	20
4.1.2 A fundamental law of electrodynamics . . . . .	21
4.1.3 The consistency problem . . . . .	21
4.2 Einstein solution: a reflection about time . . . . .	21
4.2.1 Simultaneity . . . . .	22
4.3 Lorentz transformations . . . . .	23
4.3.1 The algebraic derivation . . . . .	23
4.4 Synopsis . . . . .	25
<b>5 Lecture 5</b>	<b>27</b>
5.1 The group of Lorentz transformations . . . . .	27
5.1.1 2-dimensional case . . . . .	27
5.1.2 A comment about the 4-dimensional case . . . . .	30
5.2 Synopsis . . . . .	31
<b>6 Lecture 6</b>	<b>33</b>
6.1 Tensors - 1 - . . . . .	33
6.1.1 Tensor product . . . . .	33
6.1.2 Properties of tensor product . . . . .	35
<b>7 Lecture 7</b>	<b>39</b>
7.1 Tensors - 2 - . . . . .	39
7.1.1 Additional properties of tensor product . . . . .	39
7.1.2 Isomorphism with multilinear transformations . . . . .	41
7.1.3 Tensors and components . . . . .	42
7.2 Synopsis . . . . .	43
<b>8 Lecture 8</b>	<b>45</b>
8.1 Some reminders of topology . . . . .	45
8.2 Some reminders of differential geometry . . . . .	46
8.2.1 Vector bundles and sections . . . . .	47
8.2.2 Partition of unity . . . . .	47
8.3 Tensors - 3 - . . . . .	49
8.3.1 Tensors (Tensor Fields) on Manifolds . . . . .	49
<b>9 Lecture 9</b>	<b>55</b>
9.1 Some algebraic preliminaries . . . . .	55
9.1.1 Scalar products on a vector space . . . . .	55
9.2 Riemannian (Lorentzian) geometry - 1 - . . . . .	57
9.2.1 Riemannian and Lorentzian manifolds . . . . .	57
9.3 Connections on manifolds - 1 - . . . . .	58
9.3.1 Connections and symmetric connections . . . . .	59
<b>10 Lecture 10</b>	<b>61</b>
10.1 Connections on manifolds - 2 - . . . . .	61
10.1.1 Characterization of symmetric connections . . . . .	61
10.1.2 Smooth curves and covariant derivative along a curve . . . . .	63

---

---

<b>11 Lecture 11</b>	<b>67</b>
11.1 Tensors - 4 - . . . . .	67
11.1.1 A few more concepts about tensors . . . . .	67
11.2 Connections on manifolds - 3 - . . . . .	68
11.2.1 Parallel vector fields and parallel translation . . . . .	68
11.2.2 Extension of covariant derivative to tensors . . . . .	69
<b>12 Lecture 12</b>	<b>71</b>
12.1 Connections on manifolds - 4 - . . . . .	71
12.1.1 Component expression of the covariant derivative . . . . .	71
12.2 Curvature - 1 - . . . . .	74
12.2.1 Definition of the Riemann tensor and a basic property . .	74
<b>13 Lecture 13</b>	<b>77</b>
13.1 Curvature - 2 - . . . . .	77
13.1.1 Components of the Riemann tensor, symmetries and Ricci tensor . . . . .	77
<b>14 Lecture 14</b>	<b>81</b>
14.1 More about curves on manifolds - 1 - . . . . .	81
14.1.1 Autoparallel curves and the exponential map . . . . .	81
<b>15 Lecture 15</b>	<b>85</b>
15.1 Riemannian (Lorentzian) geometry - 2 - . . . . .	85
15.1.1 Interplay between connection and metric . . . . .	85
15.2 Curvature - 2 - . . . . .	89
15.2.1 Curvature on Riemannian (Lorentzian) Manifolds . . . . .	89
<b>16 Lecture 16</b>	<b>91</b>
16.1 Notation, Greek indices, . . . . .	91
16.2 General Relativity . . . . .	93
16.2.1 Problems of the special theory . . . . .	93
16.2.2 Einstein's elevator thought experiment . . . . .	96
16.3 Synopsis and a word of caution . . . . .	97
<b>17 Lecture 17</b>	<b>99</b>
17.1 More about curves on manifolds - 2 - . . . . .	99
17.1.1 Geodesics . . . . .	100
17.2 Mathematical formulation of General Relativity . . . . .	100
17.3 The classical limit of the geodesic equation . . . . .	102
<b>18 Lecture 18</b>	<b>105</b>
18.1 Einstein Equations . . . . .	105
18.1.1 Action for the gravitational field . . . . .	105
18.1.2 Variational principle . . . . .	105
18.1.3 The Lagrangian density . . . . .	105
18.1.4 Derivations of Einstein equations in vacuo . . . . .	106

---

<b>19 Lecture 19</b>	<b>109</b>
19.1 Properties of Einstein's field equation . . . . .	109
19.2 Physical meaning of the metric fields . . . . .	111
19.2.1 Time measurements . . . . .	111
19.2.2 Space measurements . . . . .	112
19.2.3 Clock synchronization . . . . .	113
19.3 More about the classical limit . . . . .	113
19.4 Synopsis . . . . .	114
<b>20 Lecture 20</b>	<b>115</b>
20.1 Conservation of energy in classical mechanics . . . . .	115
20.2 Conservation laws in a special relativistic field theory . . . . .	116
20.3 Conservation laws and general covariance . . . . .	117
20.4 Einstein equations . . . . .	121
<b>21 Problems</b>	<b>123</b>
<b>A Parallelism and Autoparallelism</b>	<b>125</b>
A.1 Autoparallel curves . . . . .	125

# List of Figures

8.1	Typical example of a non-Hausdorff topological space. . . . .	47
8.2	Vector bundle. . . . .	48
8.3	Section of a vector bundle. . . . .	48
9.1	Timelike, spacelike and null vectors. . . . .	56
14.1	Exponential of a vector. . . . .	82



# **List of Tables**



# List of Definitions

3.1	Field . . . . .	13
3.2	Functional of fields . . . . .	13
3.3	Field fluctuation (or variation) . . . . .	13
3.4	Finite variation of a functional . . . . .	14
3.5	. . . . .	14
3.6	Extremal of a functional . . . . .	14
3.7	Euler-Lagrange equations . . . . .	14
6.1	Tensor product . . . . .	33
6.2	Universal factorization property . . . . .	34
8.1	Topology and open sets . . . . .	45
8.2	Topological space . . . . .	45
8.3	Neighborhood . . . . .	45
8.4	Cover . . . . .	45
8.5	Subcover . . . . .	45
8.6	Refinement . . . . .	46
8.7	Open cover . . . . .	46
8.8	Locally finite open cover . . . . .	46
8.9	Compact topological space . . . . .	46
8.10	Paracompact topological space . . . . .	46
8.11	Hausdorff topological space . . . . .	46
8.12	Vector bundle . . . . .	47
8.13	Section of a vector bundle . . . . .	47
8.14	Differentiable partition of unity . . . . .	47
8.15	Tensor bundle of the $(r, s)$ type . . . . .	49
8.16	Smooth tensor field . . . . .	50
8.17	Tangent and Cotangent bundle . . . . .	50
8.18	Vector fields and 1-form fields . . . . .	50
8.19	Line element field . . . . .	50
8.20	Lie Brackets . . . . .	52
9.1	Scalar product . . . . .	55
9.2	Signature and Lorentzian metric . . . . .	56
9.3	Timelike, spacelike and null vectors . . . . .	56
9.4	Riemannian metric . . . . .	57
9.5	Lorentzian metric . . . . .	57
9.6	Isometry between manifolds . . . . .	58
9.7	Connection at $\mathbf{m} \in \mathcal{M}$ . . . . .	59
9.8	Connection on a manifold . . . . .	60
9.9	Symmetric connection . . . . .	60
9.10	Connection in coordinates . . . . .	60

10.1 Smooth curve on a manifold . . . . .	63
11.1 Tensor algebra . . . . .	67
11.2 Contractions of a tensor . . . . .	67
11.3 Parallel vector field along a curve . . . . .	68
11.4 Covariant derivative of vector fields . . . . .	69
11.5 Extension of covariant derivative . . . . .	69
12.1 Riemann curvature tensor . . . . .	74
13.1 Ricci tensor . . . . .	79
14.1 Autoparallel . . . . .	81
14.2 . . . . .	82
15.1 Compatibility condition . . . . .	85
15.2 Ricci scalar . . . . .	90
15.3 Einstein tensor . . . . .	90
17.1 Timelike, spacelike, null vectors on a manifold . . . . .	99
17.2 Local character of a curve at a point . . . . .	99
17.3 Global character of a curve . . . . .	99
17.4 Length of a curve . . . . .	100
17.5 . . . . .	100
20.1 Stress Energy Tensor . . . . .	117

# List of Propositions

3.1	Conditions for an extremal . . . . .	15
4.1	Galilean law of composition of velocities . . . . .	20
6.1	Tensor product: universal factorization . . . . .	34
6.2	Isomorphism of $V \otimes W$ into $W \otimes V$ . . . . .	35
6.3	Isomorphism of $\mathbb{F} \otimes U$ onto $U$ . . . . .	36
6.4	Isomorphism of $(U \otimes V) \otimes W$ onto $U \otimes (V \otimes W)$ . . . . .	36
6.5	Tensor product of functions . . . . .	37
7.1	Distributive properties of $\otimes$ with respect to $\oplus$ . . . . .	39
7.2	Basis of tensor product . . . . .	39
7.3	Tensor product and linear applications . . . . .	39
7.4	Tensor product and duals . . . . .	40
7.5	Tensor product and linear mappings . . . . .	41
8.1	Existence of partition of unity . . . . .	49
8.2	Characterization of smooth tensor fields . . . . .	50
8.3	Characterization of smooth vector fields . . . . .	51
8.4	Properties of the Lie Brackets . . . . .	52
9.1	Existence of Riemannian metric . . . . .	57
9.2	Existence of Lorentzian metric . . . . .	58
10.1	Characterization of symmetric connections . . . . .	61
10.2	Covariant derivative along a curve . . . . .	64
11.1	Characterization of parallel vector field . . . . .	68
11.2	Existence of parallel vector fields . . . . .	68
11.3	Parallel translation is an isomorphism . . . . .	68
12.1	Riemann tensor and covariant derivatives . . . . .	75
13.1	Riemann tensor and coordinate basis . . . . .	77
13.2	Properties of the Riemann tensor . . . . .	78
14.1	Autoparallelism equation . . . . .	81
14.2	. . . . .	82
15.1	I characterization of compatible connections . . . . .	85
15.2	II characterization of compatible connections . . . . .	87
15.3	$\exists!$ symmetric compatible connection . . . . .	87
15.4	More symmetries of the Riemann tensor . . . . .	89
15.5	Symmetries of the Ricci tensor . . . . .	89
15.6	Differential identities of curvature tensors . . . . .	90
17.1	Character of geodesics . . . . .	100
17.2	. . . . .	100
20.1	Local conservation laws . . . . .	117



# List of Notations

7.1 . . . . .	41
8.1 Particular cases of bundles and spaces of fields . . . . .	50
12.1 Covariant derivative components . . . . .	72
15.1 Compatible Symmetric Covariant Derivative . . . . .	89

