Chapter 8

General Relativity

8.1 Einstein's Equations: intuitive derivation

8.2 Einstein's Equations: structure

8.3 Einstein's Equations, geodesics and the classical limit

8.4 Basics of Causal Structure

We have seen that a solution of Einstein's equations of General Relativity is a metric (tensor), $g_{\mu\nu}$, associated with a 4-dimensional manifold (spacetime), $\mathscr{M}$: this is way we adhere to the convention of denoting spacetime with a couple $(\mathscr{M}, \boldsymbol{g})$.

8.4.1 Characters of vectors and curves

We have also stressed more than once that at any given point $\mathtt{m} \in \mathscr{M}$ (event) the metric defines the result of the scalar product between any two vectors in the *tangent space* $\mathscr{M}_{\mathtt{m}}$ to $\mathscr{M}$ at p and thus also the norm of any single vector in $\mathscr{M}_{\mathtt{m}}$. According to the sign of the norm, we also have classified vectors as TIMELIKE (future or past directed), NULL (future or past directed) or SPACELIKE.

A few remarks on time-orientability

Since a tangent vector $\boldsymbol{v}_0$ at a given event $\mathtt{p}_0 \in \mathscr{M}$ can be taught as the velocity vector of a test particle passing through $\mathtt{p}_0 = \gamma(t_0)$ at a given instant of time t_0 of its evolution along the trajectory $\gamma(t)$, the curve $\gamma(t)$ can be identified as TIMELIKE (PAST or FUTURE DIRECTED), NULL (PAST or FUTURE DIRECTED) or SPACELIKE at a given event as follows.

Definition 8.1 (Character of a curve at a point)
A curve $\gamma(t)$ is timelike *(past or future directed),* null *(past or future directed) or* spacelike *at $\mathtt{p}_0 = \gamma(t_0)$, if its tangent vector $\dot{\boldsymbol{\gamma}}(t_0)$ at $\mathtt{p}_0$ is timelike (past or future directed) or null (past or future directed) or spacelike.*

Since often we need curves which maintain their character, the definition above can be restricted.

Definition 8.2 (Global character of curves)
A curve $\gamma_{\rm T}$ which is timelike at every event is a timelike curve*, a curve $\gamma_{\rm L}$ which is null at every event is a* null curve*, a curve $\gamma_{\rm S}$ which is spacelike at every event is a* spacelike curve *and finally a curve $\gamma_{\rm C}$ which is timelike or null at every event (i.e. which is nowhere spacelike) is a* causal curve*.*

These last definitions are of a global character (whereas those associated with the behaviour at a single spacetime event are local ones).

8.4.2 Causal relations between events

With the concepts introduced so far it is possible to precisely define the causal relation between two arbitrary spacetime events and particular definitions are reserved for the sets of events which are in a one of the above defined relations with a chosen reference event.

Definition 8.3 (Chronological past and future of an event)
In particular given an event $\mathtt{p} \in \mathscr{M}$, the chronological future *(resp.* past*) of* $\mathtt{p}$*, is the set $I^+(\mathtt{p})$ (resp. $I^-(\mathtt{p})$) of all points that can be reached by a future (resp. past) directed timelike curve starting at $\mathtt{p}$.*

In an analogous way the causal past and future of an event can be defined.

Definition 8.4 (Causal past and future of an event)
The causal future *(resp.* past*) of $\mathtt{p} \in \mathscr{M}$, is the set $J^+(\mathtt{p})$ (resp. $J^-(\mathtt{p})$) of all points that can be reached by a future (resp. past) directed causal curve starting at $\mathtt{p}$.*

If instead of focusing our attention on a single event we consider a subset $U \subset \mathscr{M}$ the above definitions are generalized in a natural way by the union of the corresponding sets for all $\mathtt{q} \in U$.

Definition 8.5 (Chronological and causal future (past) of a set)
The sets

$$I^{\pm}(\mathcal{U}) \equiv \bigcup_{q \in \mathcal{U}} I^{\pm}(q) \quad \text{and} \quad J^{\pm}(\mathcal{U}) \equiv \bigcup_{q \in \mathcal{U}} J^{\pm}(q)$$

are the chronological *and* causal $\begin{smallmatrix}\text{future}\\ \text{past}\end{smallmatrix}$ of $\mathcal{U}$ *respectively.*

8.4.3 Causality, initial conditions, domains of dependence and Cauchy surfaces

With the above definitions we are now in a position to rigorously speak about the structure of spacetime with respect to given (sets of) events. Of course, many pathological situation could occur in a general case, i.e. we could be easily confronted with spacetime structures which do not satisfy some natural properties which we understand under the word "causality" in its intuitive meaning. Even if this structures can be quite stimulating from the point of view of present day research in what follows we will restrict our attention to spacetime manifolds which are commonly referred as strongly causal spacetimes.

Definition 8.6 (Strongly causal spacetime)
A spacetime $\mathscr{M}$ is strongly causal *if given an arbitrarily chosen event $p \in \mathscr{M}$ for each $U \subset \mathscr{M}$ open neighborhood of p there exist another open neighborhood of p, $V \subset U$, such that no casual curve intersects it more than once.*

Incidentally, note that spacetimes in which there are closed causal curves, so that events can influence their past, do not satisfy this condition: strong causality, at least from this point of view, seems thus a sensible physical requirement.

We now remember what we have already discussed when confronted with the structure of Einstein's equations: we have seen how and on which quantities some suitable initial conditions have to be imposed. It is also interesting to understand how it is possible to characterize in a convenient way subsets of spacetime on which these boundary conditions are to be given. These subsets should be considered adequate as "generators" from the viewpoint of causal relations among events, i.e. from the viewpoint of spacetime structure.

As a preliminary consideration we note that from one of these subsets of $\mathscr{M}$ we would like to be able to determine the structure of spacetime for an arbitrary time interval in the future. This is the reason for which we need a further characterization of the definition of causal curve given above.

Definition 8.7 (Inextendible causal curve)
A causal curve γ_{C} is called future *(resp.* past*)* inextendible *if it is impossible to find an event $p \in \mathscr{M}$ such that for all $U \subset \mathscr{M}$, U neighborhood of p, there exist a t' such that $\gamma_{\mathrm{C}}(t) \in U$ for all $t > t'$ (resp $t < t'$)*

In more concrete words, this means that γ_{C} has no *future* (resp. *past*) *endpoint.*

Then we observe that initial conditions are usually given at a fixed "initial time", i.e. on a subset of events which are simultaneous in some sense. Since from the causal viewpoint simultaneity can be understood as the absence of any kind of time ordering relation, we will single subsets of $\mathscr{M}$ with this property in the following definition.

Definition 8.8 (Achronal sets)
Subsets, $\mathscr{A} \subset \mathscr{M}$ of spacetime characterized by the property that $\mathscr{A} \cap I^{+}(\mathscr{A}) = \emptyset$ are called achronal sets.

The physical meaning of this definition is that in these sets no event is in the future of another one and thus no concept of future (or past) causal relationship can be established among the events in the set.

If we add some particular events to an achronal set, i.e. if we make an achronal set $\mathscr{A}$ a little bit larger by adding some particular points, then the obtained set could contain some events causally related to some other events in the set, so that the new set would be not achronal any more. This particular events are thus singled out by the following definition.

Definition 8.9 (Edge of an achronal set)
Let $\mathscr{A}$ be an achronal set. The edge of $\mathscr{A}$, $E(\mathscr{A})$, is the subsets of all the events $p \in \mathscr{A}$ such that every neighborhood of p, $U \subset \mathscr{M}$, contains at least a point $p_{+} \in I^{+}(p)$, a point $p_{-} \in I^{-}(p)$ and a timelike curve γ_{T}, connecting p_{-} and p_{+}, such that $\gamma_{\mathrm{T}} \cap \mathscr{A} = \emptyset$.

We can now merge the last two concepts that we have defined.

Definition 8.10 (Domains of dependence)
Let $\mathscr{A}$ be a closed achronal set. The set $D^+(\mathscr{A})$ (resp. $D^-(\mathscr{A})$) of all spacetime events p *such that every past (resp. future) inextendible causal curve passing through* p *intersects $\mathscr{A}$ is called the* future *(resp.* past*)* domain of dependence of $\mathscr{A}$*. The set $D(\mathscr{A}) = D^+(\mathscr{A}) \cup D^-(\mathscr{A})$, union of the past and of the future domains of dependence is the* domain of dependence *of $\mathscr{A}$.*

From the physical point of view all events in $D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$) can be considered as effects (resp. causes) of the events in $\mathcal{A}$ and a special important case is the one in which $D(\mathscr{A})$ is all of $\mathscr{M}$, since starting from the "simultaneous" events of the achronal set, the future evolution of all spacetime can be predicted (since all the events that constitute it are causally determined by the ones in $\mathscr{A}$). Moreover we also have a complete knowledge of its past history. This motivates the following definition.

Definition 8.11 (Cauchy surface and global hyperbolicity)
Let $\mathscr{A} \subset \mathscr{M}$ be an achronal set such that $D(\mathscr{A}) = \mathscr{M}$. Then $\mathscr{A}$ is called a Cauchy surface *(we instead use the denomination* partial Cauchy surface *for a closed achronal set without edge). A spacetime $\mathscr{M}$ which admits a Cauchy surface is called* globally hyperbolic*.*

We observe, without proof, that a Cauchy surface is an achronal set with empty edge.

8.4.4 Asymptotic structure

Till now we have set up some "causal language" in terms of which we can understand the structure of spacetime as defined by the relations among the events, which are its constituents. It is often useful have the possibility to compare the results obtained in General Relativity with those of Classical Theory, for example Newtonian gravitation. This is usually understood as the limit of weak gravity, in which the spacetime structure is nearly that of flat Minkowski space. It can, of course, be too restrictive a set up in which this condition is satisfied on all spacetime, but certainly it can hold far away from the source, i.e. in what, in flat space, we consider as infinity. We will now show as it is possible to translate the concept of asymptotically flat and simple space in the language used in the previous subsection. this will also give us the possibility of properly describing a *no-escape* region. To get first an intuitive grasp of this concept we observe that if, starting from a given event p, we can find only causal curves connecting it with events in a spatially bounded proper subset of space, then we will say that we cannot escape from some region containing p. Otherwise, we will be free to have a causal connection with events that are outside any bounded proper subset of space, i.e. even with events which are at "*infinity*". It is clear that in this description infinity is considered in a very broad sense as a (part of the) boundary of spacetime, $\mathscr{M}$. This very general description of infinity will be of help to translate in the causal language of curved space the idea, typical in classical mechanics, that a body is not bound to another if it has enough energy to move far and far away, since we say in that case that it can reach infinity, which we can imagine as flat because free of source and far away from them. This motivations are behind the following definition.

Definition 8.12 (Asymptotically empty and simple spacetime)
Let us consider a strongly causal spacetime $(\mathscr{M}, \boldsymbol{g})$*. It is said to be* asymptotically empty and simple *if there exists a spacetime* $(\widetilde{\mathscr{M}}, \tilde{\boldsymbol{g}})$ *(called the* associated unphysical space*) and* $\imath$*,* $\imath : \mathscr{M} \longrightarrow \widetilde{\mathscr{M}}$*, an imbedding of* $\mathscr{M}$ *as manifold with smooth boundary* $\partial\mathscr{M}$ *in* $\widetilde{\mathscr{M}}$*, such that:*

1. *a sufficiently regular function,* Ω*, on* $\widetilde{\mathscr{M}}$ *exists with the properties that it is positive on* $\imath(\mathscr{M})$ *and* $\Omega^2 \boldsymbol{g} = \imath_*(\tilde{\boldsymbol{g}})$*: thus* $\imath$ *is a* conformal map*;*

2. $\Omega = 0$ *and* $d\Omega \neq 0$ *on* $\partial\mathscr{M}$*;*

3. *every null geodesic on* $\mathscr{M}$ *has two endpoints on* $\partial\mathscr{M}$*;*

4. $R_{\mu\nu} = 0$ *on an open neighborhood of* $\partial\mathscr{M}$ *in* $\mathscr{M} \cup \partial\mathscr{M}$ *(which is the closure* $\overline{\mathscr{M}}$ *of* $\mathscr{M}$*).*

We can thus see that an asymptotically empty and simple spacetime has a boundary that resembles the properties of infinity in Minkowski space. Moreover from the above definition, it turns out that in an asymptotically empty and simple spacetime the boundary $\partial\mathscr{M}$ is a null surface: spacetime is then in its past or in its future, so that $\partial\mathscr{M}$ actually consists of two connected components, $\mathscr{I}^+$ and $\mathscr{I}^-$, which of course are null. Moreover (with reference to the preliminary observations given above) we see that $\partial\mathscr{M}$ can also be thought to be at infinity, because the affine parameter t of each null geodesic $\gamma_L(t)$ has unbounded large values when approaching $\partial\mathscr{M}$. Physically the set $\mathscr{I}^+$ can be characterized as that connected component of $\partial\mathscr{M}$ on which null geodesics have their future endpoints, whereas on $\mathscr{I}^-$ they have their past ones. Unfortunately this very clear asymptotic structure is not general enough to cover all physically interesting cases. For this reason it is indeed convenient to generalize the concept.

Definition 8.13 (Weakly asymptotically empty and simple space)
A spacetime $(\mathscr{M}, \boldsymbol{g})$ *is called* weakly asymptotically empty and simple *if there exists an asymptotically empty and simple spacetime* $(\mathscr{M}', \boldsymbol{g}')$ *and a neighborhood* $\mathscr{N}'$ *of* $\partial\mathscr{M}'$ *in* $\mathscr{M}'$ *such that* $\mathscr{N}' \cap \mathscr{M}'$ *is isometric to an open set* $\mathscr{N} \subset \mathscr{M}$*: in this case the region* $\mathcal{N}$ *is asymptotically empty and simple.*

Of course a weakly asymptotically empty and simple spacetime can have many (even an infinite number of) asymptotically empty and simple regions.

If $(\mathscr{M}, \boldsymbol{g})$ is a weakly asymptotically empty and simple space, it is thus possible to think it as conformally imbedded as $\overline{\mathscr{M}}$ (i.e. a manifold with boundary) in a space $(\widetilde{\mathscr{M}}, \tilde{\boldsymbol{g}})$ in such a way that a neighborhood U of $\partial\mathscr{M}$ in $\widetilde{\mathscr{M}}$ is isometric to a neighborhood U' of the boundary $\partial\mathscr{M}'$ of an asymptotically empty and simple space $\mathscr{M}'$: thus $\partial\mathscr{M}$ in $\widetilde{\mathscr{M}}$ consists of the two null surfaces $\mathscr{I}^+$ and $\mathscr{I}^-$, i.e. future and past null infinity.

In an asymptotically empty and simple spacetime we will have a consistent causal structure if no singular points in spacetime can be seen from future null infinity. This motivates a further definition, with which we will merge the asymptotically empty and simple structure defined above with the ideas of causal evolution previously defined.

Definition 8.14 (Strongly future asymptotically predictable space) *Let us now consider $(\mathscr{M}, \boldsymbol{g})$, a weakly asymptotically empty and simple spacetime, and a partial Cauchy surface, $\mathscr{S}$ in $\mathscr{M}$. $(\mathscr{M}, \boldsymbol{g})$ is called* strongly future asymptotically predictable from the partial Cauchy surface *$\mathscr{S}$ if $\mathscr{I}^+$ is contained in the closure of $D^+(\mathscr{S})$ in $\widetilde{\mathscr{M}}$ and if $J^+(\mathscr{S}) \cap \bar{J}^-(\mathscr{I}^+)$ is contained in $D^+(\mathscr{S})$.*

This physically means that starting from $\mathscr{S}$ we can predict all the future until $\mathscr{I}^+$, so that as required by the above motivation there will be no singular points in spacetime which can be seen from future null infinity, and the same is true for a neighborhood of $\partial J^-(\mathscr{I}^+)$.

Moreover a strongly future asymptotically predictable space has the nice property that given a partial Cauchy surface $\mathscr{S}$ there exist a family $\mathscr{S}(\tau)$ of spacelike surfaces, homeomorphic to $\mathscr{S}$, which cover $D^+(\mathscr{S}) - \mathscr{S}$ and intersect $\mathscr{I}^+$. Each $\mathscr{S}(\tau)$ can be interpreted as a constant-time surface so that the future asyptotically predictable region can be *foliated*. Moreover the family $\mathscr{S}(\tau)$ is such that:

1. $I^+(\mathscr{S}(\tau_1)) \supset \mathscr{S}(\tau_2)$ if $\tau_1 < \tau_2$ so that, as one would expect, a spacelike surface of the family is always in the (chronological) future of surfaces at previous instants of time;

2. the edge of each spacelike surface $\mathscr{S}(\tau)$ in $\widetilde{\mathscr{M}}$ is a 2-sphere $\mathbb{S}^2(\tau)$ in $\mathscr{I}^+$;

3. at all instants of time, $\mathscr{S}(\tau) \cup \big(\mathscr{I}^+ \cap J^-(\mathbb{S}^2(\tau))\big)$ is a Cauchy surface for $D(\mathscr{S})$.

We note that in this situation if $\partial J^-(\mathscr{I}^+) \neq \emptyset$ then it has a nonempty intersection with $\mathscr{S}(\tau)$ for sufficiently large τ (for property 2. above): then there will be some events on $\mathscr{S}(\tau)$, at least for sufficiently large τ, which are not causally connected to future null infinity. We are going to use this property in the definition of a black hole.

8.4.5 Black Hole and Event Horizon

A black hole, *Loosely speaking* is a region in the spacetime structure where the pull of gravity is so strong that even light is confined inside it.

From the point of view of the Newtonian theory of gravitation, the possible existence of such regions in space(time) was first pointed out more than two centuries ago by John Mitchell [1], who realized that

> “... if the semi-diameter of a sphære of the same density with the sun were to exceed that of the sun in the proportion of 500 to 1, a body falling from an infinite height towards it, would have acquired at its surface a greater velocity than that of light, and consequently, supposing light to be attracted by the same force in proportion to its vis inertiæ, with other bodies, all light emitted from such a body would be made to return towards it, by its own proper gravity ...”.

Of course we are now in a position to give to this qualitative idea of a region were the escape velocity exceeds the speed of light (or, equivalently, the gravitational potential exceeds the square of the speed of light) a more precise, geometrical meaning in a rigorous mathematical sense. Einstein's theory of General Relativity is indeed a theory of the causal structure of spacetime in which the causal relation between events can be described by the words and concepts defined in the previous subsections, which we will apply to the black hole concept in the following definition.

Definition 8.15 (Black hole and event horizon)
Let us consider a strongly future asymptotically empty and simpl spacetime $(\mathscr{M}, \boldsymbol{g})$ *and let* $\mathscr{S}(\tau)$ *denote the family of spacelike surfaces in terms of which an asymptotically empty and simple region in it can be foliated. A* black hole *on* $\mathscr{S}(\tau)$ *is a connected component of the set*

$$\mathcal{B}(\tau) = \mathscr{S}(\tau) - J^{-}(\mathscr{I}^{+}).$$

Moreover the spacetime region

$$\partial J^{-}(\mathscr{I}^{+})$$

is called the event horizon.

This definition translates in a proper geometric language the idea of the presence in spacetime of a "*no-escape region*", which is the main intuitive property behind the black hole concept. A black hole is indeed defined as a region from which particles and light rays cannot escape to $\mathscr{I}^{+}$ and we note that in this definition it is a "hole" in the usual spatial sense, being contained in the constant-time surface $\mathscr{S}(\tau)$. The event horizon is an achronal set, the boundary of the region from which particles and light rays can escape to infinity, and is generated by null geodesic segments with possibly past endpoints but no future endpoints.