

Geometry and SpaceTime Structure

Stefano Ansoldi

Dipartimento di Fisica Teorica

Università degli Studi di Trieste

Corso di Laurea in Fisica

©2004

Copyright by Stefano Ansoldi and the University of Udine, 2004.

All Italian and international copyrights reserved
for all original material presented in this course and lecture notes
through any medium, including lecture or print.

Individuals are prohibited from being paid for taking, selling,
or otherwise transferring for value, this material or part of it
without the express written permission of Stefano Ansoldi.

Individuals are prohibited from distributing by any means
this material or part of it
without the express written permission of Stefano Ansoldi.

Individuals are granted the right of using this material
for personal, educational purposes only
without the express written permission of Stefano Ansoldi.

Contents

Contents	i
List of Figures	v
List of Tables	vii
List of Definitions	ix
List of Propositions	xiii
List of Notations	xv
II Acknowledgments	xvii
1 Preliminaries	1
1.1 Linear Algebra preliminaries	1
1.2 Structures over a vector space	1
1.2.1 Exterior algebra	1
1.2.2 Tensor algebra	2
1.2.3 Orientation	9
1.2.4 Scalar product	10
1.3 Topology preliminaries	11
2 Exercises	15
3 Differential Geometry	17
3.1 Differentiable Manifold	17
3.2 Maps on Manifolds	19
3.3 Partition of unity	20
3.4 Tangent Space	20
3.5 Cotangent Space and the differential	22
3.6 Forms and Tensor at a point	25
3.7 Bundles	26
3.8 Fields	29
3.9 Orientation on manifolds	33
3.10 Exterior differential	36
3.11 Maps between manifolds	36
3.12 Vector fields and integral curves	39
3.13 Lie derivative	42

3.14	Integration on manifolds	45
3.15	Riemannian and Lorentzian manifolds	47
3.16	Connection and covariant derivative	49
3.17	Interplay between connection and metric	54
3.18	Geodesics	58
3.19	Curvature	61
4	Exercises	67
4.1	Connection and Covariant Derivative	67
5	Relativity in Einstein's Words	75
5.1	Reflections on spacetime	75
5.1.1	Space and time in pre-relativistic physics	75
5.1.2	Classical mechanics and its framework	75
5.1.3	Electrodynamics	76
5.1.4	The consistency problem	77
5.2	Einstein solution: a reflection about time	77
5.2.1	Simultaneity	77
5.2.2	Relativity of simultaneity	79
5.2.3	Relativity of distance	79
5.3	Special Relativity	79
5.4	Lorentz transformations	79
5.4.1	The algebraic derivation	79
5.4.2	Space and Time in relativistic physics	81
5.4.3	The definition of Time: Relativity of Simultaneity	82
5.4.4	Limits of Special Relativity	82
5.4.5	Principle of General Covariance	82
5.4.6	The role of Gravity: Einstein's Elevator	82
5.4.7	General Relativity Principle and Gravitation	82
5.4.8	General Relativity	82
6	Special Relativity	83
6.1	The group of Lorentz transformations	83
6.1.1	2-dimensional case	83
6.2	Accelerated Observers in Minkowski spacetime	86
7	Special Relativity: Problems	93
7.1	Kinematics	93
8	Stress Energy Tensor	105
8.1	Conservation of energy in classical mechanics	105
8.2	Conservation laws in a special relativistic field theory	106
9	General Relativity: Problems	109
9.1	109

10 General Relativity	111
10.1 Einstein's Equations: intuitive derivation	111
10.2 Einstein's Equations: structure	112
10.3 Einstein's Equations, geodesics, classical limit	112
10.4 Basics of Causal Structure	115
10.4.1 Characters of vectors and curves	115
10.4.2 Causal relations between events	115
10.4.3 Causality, initial conditions, domains of dependence, Cauchy surfaces	116
10.4.4 Asymptotic structure	118
10.4.5 Black Hole and Event Horizon	120
11 General Relativity: Problems	123

List of Figures

1.1	Timelike, spacelike and null vectors.	12
1.2	Typical example of a non-Hausdorff topological space.	13
3.1	Coordinate maps and function on a locally euclidean Hausdorff space.	18
3.2	Compatibility condition for a differentiable structure.	18
3.3	Differentiable function on a manifold.	19
3.4	Differentiable map between manifolds.	19
3.5	Differential of a map between manifolds.	23
3.6	Differential of a function.	24
3.7	Vector bundle.	27
3.8	Section of a vector bundle.	27
3.9	Local triviality.	30
3.10	Parallelizable manifold.	30
3.11	Induced orientation on the boundary.	36
3.12	Pullback.	37
3.13	Pullback and exterior differentiation.	38
3.14	Pullback and composition.	38
3.15	Integral curves of a vector field.	40
3.16	Flow associated with a vector field.	42
3.17	Lie derivative of a vector field.	42
3.18	Component expression of the Lie derivative.	44
3.19	Exponential of a vector.	60
6.1	Red-shift between fundamental observers.	89
6.2	Red-shift by a stationary observer.	90
7.1	World lines and surfaces of simultaneity of observer in relative translational motion with constant velocity.	99

List of Tables

List of Definitions

1.1	k -linear alternating maps	1
1.2	Exterior product in $\Lambda^k(V)$	2
1.3	Graßmann Algebra of V	2
1.4	Tensor product	2
1.5	Universal factorization property	3
1.6	Tensors on V	8
1.7	Tensor algebra	9
1.8	Symmetrized tensor	9
1.9	Antisymmetrized tensor	9
1.10	Orientation on V	9
1.11	Scalar product	10
1.12	Signature and Lorentzian metric	11
1.13	Timelike, spacelike and null vectors	11
1.14	Topology and open sets	11
1.15	Topological space	11
1.16	Neighborhood	12
1.17	Cover	12
1.18	Subcover	12
1.19	Refinement	12
1.20	Open cover	12
1.21	Locally finite open cover	12
1.22	Compact topological space	12
1.23	Paracompact topological space	12
1.24	Hausdorff topological space	13
3.1	Locally Euclidean Hausdorff space	17
3.2	Differentiable Structure	17
3.3	Differentiable Manifold	17
3.4	C^∞ function on a manifold	19
3.5	C^∞ map between manifolds	19
3.6	Smooth curve on a manifold	19
3.7	Differentiable partition of unity	20
3.8	$C^\infty(\mathcal{M}, \mathfrak{m}, \mathbb{R})$	20
3.9	Germ of functions around $\mathfrak{m} \in \mathcal{M}$	20
3.10	Tangent vector at $\mathfrak{m} \in \mathcal{M}$	21
3.11	Tangent space at $\mathfrak{m} \in \mathcal{M}$	21
3.12	Cotangent vector at $\mathfrak{m} \in \mathcal{M}$	22
3.13	Cotangent space at $\mathfrak{m} \in \mathcal{M}$	22
3.14	Differential of a function between manifolds	22
3.15	Form at $\mathfrak{m} \in \mathcal{M}$	25

3.16	Tensor at $\mathfrak{m} \in \mathcal{M}$	26
3.17	Vector bundle	26
3.18	Section of a vector bundle	26
3.19	Tangent bundle	26
3.20	Cotangent bundle	26
3.21	Tensor bundle of the (r, s) type	28
3.22	Exterior k -bundle	28
3.23	Exterior bundle	28
3.24	Parallelizable manifold	29
3.25	Smooth vector field	29
3.26	Line element field	31
3.27	Smooth 1-form field	32
3.28	Smooth tensor field	32
3.29	Smooth k -form field	32
3.30	Smooth form field	32
3.31	Smooth vector field along a curve	33
3.32	Orientation on a manifold	33
3.33	Regular domains and outer vectors	35
3.34	Boundary of manifold	35
3.35	Orientation of the boundary	36
3.36	Pullback	36
3.37	Tensor maps induced by diffeomorphisms	39
3.38	Lie Brackets	39
3.39	Integral curves of a vector field	39
3.40	Flow associated to a vector field	41
3.41	Lie derivative of a vector field	42
3.42	Lie derivative of a 1-form field	42
3.43	Integral of an m -form (local)	45
3.44	Integral of an m -form (global)	46
3.45	Riemannian metric	47
3.46	Lorentzian metric	48
3.47	Isometry between manifolds	49
3.48	Connection at $\mathfrak{m} \in \mathcal{M}$	49
3.49	Connection on a manifold	49
3.50	Symmetric connection	49
3.51	Connection in coordinates	50
3.52	Parallel vector field along a curve	54
3.53	Compatibility condition	54
3.54	Geodesic	58
3.55		59
3.56	Riemann curvature tensor	61
3.57	Ricci tensor	65
3.58	Ricci scalar	65
3.59	Einstein tensor	66
8.1	Stress Energy Tensor	107
10.1	Character of a curve at a point	115
10.2	Global character of curves	115
10.3	Chronological past and future of an event	116
10.4	Causal past and future of an event	116
10.5	Chronological and causal future (past) of a set	116

10.6 Strongly causal spacetime	116
10.7 Inextendible causal curve	117
10.8 Achronal sets	117
10.9 Edge of an achronal set	117
10.10 Domains of dependence	117
10.11 Cauchy surface and global hyperbolicity	117
10.12 Asymptotically empty and simple spacetime	118
10.13 Weakly asymptotically empty and simple space	119
10.14 Strongly future asymptotically predictable space	119
10.15 Black hole and event horizon	120

List of Propositions

1.1	Vector space structure of $\Lambda^k(V)$	1
1.2	Basis of $\Lambda^k(V)$	1
1.3	Universal factorization property of the tensor product	3
1.4	Isomorphism of $V \otimes W$ into $W \otimes V$	4
1.5	Isomorphism of $\mathbb{F} \otimes U$ onto U	5
1.6	Isomorphism of $(U \otimes V) \otimes W$ onto $U \otimes (V \otimes W)$	5
1.7	Tensor product of functions	6
1.8	Distributive properties of \otimes with respects to $+$	6
1.9	Basis of tensor product	6
1.10	6
1.11	Tensor product and duals	7
1.12	Tensor product and linear mappings	8
1.13	Vector space structure of $T_s^r(V)$	8
1.14	Algebra structure of $T_s^r(V)$	8
1.15	Choice of an orientation of V	10
3.1	Existence of partition of unity	20
3.2	Dimension and coordinate basis of \mathcal{M}_m	21
3.3	Coordinate representation of $df _m$	23
3.4	Coordinate basis in \mathcal{M}_m^*	24
3.5	Coordinate expression of the differential	25
3.6	Vector bundles as differentiable manifolds	28
3.7	Characterization of vector fields	31
3.8	Characterization of smooth tensor fields	32
3.9	Characterization of smooth k -form fields	32
3.10	Characterization of smooth form fields	33
3.11	Differential as a 1-form field	33
3.12	Existence of line element fields	33
3.13	Characterization of orientable manifolds	34
3.14	Exterior differentiation	36
3.15	Properties of the pullback	37
3.16	Properties of the Lie Brackets	39
3.17	Equation satisfied by integral curves	40
3.18	Existence of integral curves	41
3.19	Properties of the Lie derivative	42
3.20	Lie derivative of arbitrary tensors	43
3.21	Component expression of the Lie derivative	43
3.22	Local expression of the integral	45
3.23	Coordinate independence of the local integral	45

3.24	Independence from the partition of unity choice	47
3.25	Stokes theorem	47
3.26	Existence of Riemannian metric	48
3.27	Existence of Lorentzian metric	49
3.28	Characterization of symmetric connections	50
3.29	Covariant derivative along a curve	52
3.30	Characterization of parallel vector field	54
3.31	Existence of parallel vector fields	54
3.32	Parallel translation is an isomorphism	54
3.33	Characterization of compatible connections: I	54
3.34	Characterization of compatible connections: II	56
3.35	Uniqueness of symmetric compatible connection	57
3.36	Geodesic equation	59
3.37	59
3.38	Riemann tensor and covariant derivatives	62
3.39	Riemann tensor and coordinate basis	63
3.40	Properties of the Riemann tensor	64
3.41	Additional symmetries of the Riemann tensor	65
3.42	Symmetries of the Ricci tensor	65
3.43	Differential identities of curvature tensors	66
5.1	Galilean law of composition of velocities	76
8.1	Local conservation laws	107

List of Notations

1.1	7
3.1	Compatible Symmetric Covariant Derivative	58

