Geometry and SpaceTime Structure

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Algebraic premises.

Structures on vector spaces:

1. basic notions about k-linear alternating maps over vector spaces; basis and dimension of the vector space of k-linear alternating forms; some canonical isomorphisms; exterior product of forms; exterior algebra over vector spaces;
2. basic notions about the tensor algebra over vector spaces; dimension and basis of the space of tensors of the (r, s) type; symmetric and anti-symmetric tensors; symmetrization ed anti-symmetrization of tensors; forms as totally anti-symmetric tensors;
3. orientation and choice of an orientation;
4. scalar product.

Basic notions of differential geometry.

1. Basic notions about topological spaces, open sets, topology.
   Coverings, open coverings, locally finite open coverings, refinements; compact and paracompact topological spaces.
2. Differentiable structure and differentiable manifold; (differentiable) functions and maps on/between manifolds; (differentiable) curves on manifolds.
3. Differentiable partition of unity; existence of differentiable partitions of unity (statement without proof).
4. Tangent vectors and tangent space; differential of a map at a point, cotangent vectors and cotangent space; coordinate basis in tangent and cotangent space; components of tangent and cotangent vectors; basis change in tangent and cotangent spaces and transformation law for the components; coordinate change and change in associated coordinate basis in tangent and cotangent spaces; transformation law for the components.
5. Space of k-forms at a point; basis change and transformation law for components.
   Space of tensors at a point; basis change and transformation law for components.
6. **Structures on manifolds:**
   (a) vector bundles, tangent bundle, cotangent bundle, exterior algebra, tensor bundle; bundles as differentiable manifold and their dimension; meaning of the local triviality of bundle; parallelizable manifold; sections of bundles;
   (b) vector fields on an open set and along a curve; differentiable vector fields; coordinate expression of a vector field in a given coordinate system; characterization of differentiable vector fields;

space of differentiable vector fields on a manifold; integral curve of a vector field, existence and uniqueness of integral curves; basic concepts concerning the flow associated with a vector field (definition without proof);

(c) form/tensor fields over an open set and along a curve; coordinate expression of a form/tensor field; differentiable form/tensor fields, characterization of differentiable form/tensor fields; space of differentiable form/tensor fields on a manifold; the differential of a map as a 1-form field on the manifold.

7. Exterior derivative: existence and uniqueness (statement without proof), properties.
8. Maps between manifolds and associate maps between corresponding structures over manifolds: *pull-back* and *push-forward*; diffeomorphisms as a special case; *pull-back* and exterior derivative, properties.
9. Lie derivative of a vector field; coordinate basis expression of the Lie derivative of a vector field; generalization of the Lie derivative to forms and tensors; coordinate expression of the Lie derivative; Lie derivative of a vector field and Lie brackets.
10. Orientation of a differentiable manifold, meaning and characterization of orientation (statement without proof).
11. Integration on manifolds; local definition of the integral; local expression of the integral and its independence from the choice of coordinates; global definition if the integral by means of the partition of unity; independence of the integral from the choice of partition of unity; Stokes theorem; corollary to Stokes theorem.
12. Riemannian and Lorentzian metric; Riemannian and Lorentzian manifold; Existence of Riemannian metrics on manifolds; isometries between manifolds.
    Natural volume element on a Riemannian/Lorentzian manifold and its coordinate expression by means of the Levi-Civita tensor.
13. Connection at a point and on a manifold; connection properties and local expression by means of connection symbols; covariant derivative of a vector field and covariant derivative of a vector field along a curve; parallel vector fields along a curve; compatibility condition for the connection on a Riemannian/Lorentzian manifold; necessary and sufficient condition for the compatibility of a connection on a Riemannian/Lorentzian manifold; symmetric connection; characterization of a symmetric connection in a coordinate basis and properties of the associated connection symbols; existence and uniqueness of the symmetric connection compatible with the Riemannian/Lorentzian metric.
    Covariant derivative of form/tensor fields.

14. Relations between covariant derivative and exterior derivative and between covariant derivative and Lie derivative on a manifold.
    Symmetries on manifolds, Killing vector fields and Killing equation.
15. Self-parallel curves and geodesics; affine reparametrization; local expression of the geodesic equation with an affine parametrization; existence and uniqueness of geodesics (statement without proof) and definition of the exponential map; differential of the exponential map; exponential map as a local diffeomorphism; basic notions of normal coordinate systems.
16. Curvature: Riemann and Ricci tensors and their properties; coordinate expression of the Riemann tensor; Riemann and Ricci tensor in the case of the unique symmetric compatible connection: additional properties, curvature scalar and Einstein tensor.

Spacetime, special relativity, general relativity.

1. About the principle of special relativity; inertial reference systems, principle of relativity, principle of the constancy of the speed of light and their experimental foundations.
   Physical meaning of geometrical propositions, operational definition of physical concepts e analysis of the concept of simultaneity from the point of view of its operative definition.
   Reconciliation of the apparent contradiction between the relativity principle in its particular form and the principle of the constancy of the speed of light: basics about Lorentz transformation equations.
2. Spacetime structure in special relativity: light cones, signals, inertial observers.
3. About the principle of general covariance; problems and ambiguities in the definition of inertial reference systems; general coordinate systems.
   The equivalence principle in its weak form and Einstein's lift *Gedanken experiment*; (*local*)equivalence between uniform gravitational fields and inertial fields; *gaussian* coordinate systems, description of physical properties by means of geometrical properties of the spacetime *continuum*; relation between the principle of general covariance, Einstein's equivalence principle and theory of the gravitational field.
   Relation between the operational definition of the concepts of space and time, the exchange of signals between observers and the causal structure of spacetime.
   Basics about experimental tests of the principles of general relativity and about its practical consequences.
4. Construction of a system of uniformly accelerated observers in Minkowski spacetime and derivation of its properties: red-shift, event horizon and causal spacetime structure for an accelerated observer in Minkowski spacetime; Rindler coordinates, line element in Rindler

coordinates and its relation with the Minkowskian line element; interpretation of Rindler spacetime in light of Einstein's lift *Gedanken experiment.*

The stress-energy tensor.

1. Lagrangian formulation for the non-relativistic dynamics of a point particle; energy conservation principle.
2. Basics of relativistic kinematics: action integral for the dynamics of a relativistic particle; variational principle with fixed boundaries and derivation of the equation of motion; variational principle with variable boundary and definition of momentum; relativistic angular momentum tensor and interpretation of its components in relation with the group of Lorentz transformations (interpreted as rotations in spacetime); relativistic center of mass.
3. Basics about the Lagrangian formulation of a field theory (emphasis about Lorentz invariant theories) from a variational principle: Euler-Lagrange equations; definition of the stress-energy tensor associated with the fields and its law of local conservation; symmetry of the stress-energy tensor and interpretation of its components; conservation laws in integral form and relativistic constraint between flow of the energy density and momentum density.
4. Definition of the stress-energy tensor for a field theory in presence of general covariance; local form of the conservation law; interpretation of the local conservation law and invariance under diffeomorphisms; conserved quantities in the presence of symmetries (Killing vectors) and analysis about integral (global) conservation laws in presence of general covariance; interpretation of the Lorentz invariant case as a particular case in view of the above considerations.

Einstein Equations.

1. *Heuristic* derivation of Einstein equations in a static spacetime by the analysis of the weak field limit.
2. Einstein equations (in two equivalent forms); non-linearity of the equations and peculiarities of their relation with sources.
   Basics of the structural analysis of Einstein equations and their characterization with respect to the associated Cauchy problem.
3. The metric tensor and the relations of its components with the properties of the gravitational field and of the particular choice of the reference system; relation between components of the metric tensor and possibility of synchronization of clocks.
4. Classical limit of Einstein equations and classical limit of the geodesic equation. Physical meaning of the connection coefficients and of the metric tensor as it results from the classical limit of Einstein equations.

Basics about the large scale structure of spacetime.

1. characterization of the type of vectors on a Lorentzian manifold; characterization of curves at a point; general characterization of curves as timelike, null, spacelike and causal.
2. Classification of geodesics and invariance of their characterization.
3. mathematical formulation of spacetime as a Lorentzian manifold; physical interpretation of geometrical entities associated with the concept of manifold: exponential map, geodesics, normal coordinate systems and their relations with the physical concepts of free motion, causal structure and equivalence principle.
4. Basics about the definition of black hole from the point of view of causal structure (introduction and definition of the following fundamental concepts and their physical meaning, without proofs):
    - future/past directed timelike, spacelike null and causal curves; past/future endpoint of a causal curve; past/future inextendible causal curves;
    - chronological and causal past/future of an event and of a set of events;
    - strongly causal spacetime;
    - achronal set and edge of an achronal set; past/future domain of dependence of a closed achronal set; domain of dependence of a closed achronal set; Cauchy surface and partial Cauchy surface; globally hyperbolic spacetime;
    - asymptotically empty and simple spacetime and associate unphysical space; properties of an asymptotically empty and simple spacetime: asymptotically null past/future infinity and characterization in terms of geodesics; weakly asymptotically empty and simple spacetime;
    - strongly future asymptotically predictable spacetime starting from a partial Cauchy surface and its properties;
    - black hole;
    - event horizon.