Chapter 21

Problems

Problem 21.1 (Change of coordinates and tensor components)
Let us consider $\boldsymbol{T} \in T^2_1(V)$ *with* V *a vector space of dimension* n*. Let us fix a basis* $\{\boldsymbol{e}_i\}_{i=1,\dots,n}$ *in* V *and let* $\{\boldsymbol{E}^i\}_{i=1,\dots,n}$ *be the dual basis in* V^**. Let us consider a change of basis in* V *and let* $\{\tilde{\boldsymbol{e}}_i\}_{i=1,\dots,n}$ *and* $\{\tilde{\boldsymbol{E}}^i\}_{i=1,\dots,n}$ *be the new basis of* V *and of* V^**. Let us denote the change of basis as* $\tilde{\boldsymbol{e}}_i = \Lambda_i{}^j \boldsymbol{e}_j$*. Write the relation between* T^{ij}_k*, the components of* $\boldsymbol{T}$ *in the first basis, and* $\tilde{T}^{ab}_c$*, the components of* $\boldsymbol{T}$ *in the second basis. Generalize this result for a general tensor* $\boldsymbol{T} \in T^r_s(V)$*.*

Problem 21.2 (Contraction of a tensor in coordinates)
Let us consider $\boldsymbol{T} \in T^2_1(V)$ *with* V *a vector space of dimension* n*. Let us fix a basis* $\{\boldsymbol{e}_i\}_{i=1,\dots,n}$ *in* V *and let* $\{\boldsymbol{E}^i\}_{i=1,\dots,n}$ *be the dual basis in* V^**. Consider the contraction* C^1_1*. Write the components of the tensor* $C^1_1\boldsymbol{T}$*. Express now the tensor* T *in a new basis* $\tilde{\boldsymbol{e}}_i = \Lambda_i{}^j \boldsymbol{e}_j$ *and in the corresponding dual basis. Write the components of the tensor* $C^1_1\boldsymbol{T}$ *in the new basis. Generalize the above result to the* C^i_j *contraction of a generic tensor* $\boldsymbol{T} \in T^r_s(\mathscr{M})$*.*

Problem 21.3 (Covariant derivative: component expression)
Let us consider a vector field $\boldsymbol{V}$*, a vector field* $\boldsymbol{W}$ *on a manifold* $\mathscr{M}$ *and a coordinate system* (U, ϕ) *associated to coordinates* $(x^1, \dots, x^m)$*.*

1. *Write the components of* $D(\boldsymbol{V}, \boldsymbol{W})$ *in the given coordinate system.*
2. *Consider a second coordinate system* (V, ψ) *associated to coordinates* $(y^1, \dots, y^m)$ *and with* $U \cap V \neq 0$*. Write then the change in the components of* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *in* $U \cap V$*. Write also the change in the components of* $D(\boldsymbol{V}, \boldsymbol{W})$ *in* $U \cap V$*.*

Do the same computations for a tensor $\boldsymbol{\omega} \in T^0_1(\mathscr{M})$ *and for a tensor* $\boldsymbol{T} \in T^1_2(\mathscr{M})$*. Generalize all the results for a generic* $\boldsymbol{T} \in T^r_s(\mathscr{M})$*.*

Solution:

Let us consider a basis $\{\boldsymbol{e}_i\}_{i=1,\dots,n}$ and the corresponding dual basis $\{\boldsymbol{E}^j\}_{j=1,\dots,n}$. By the definition of covariant derivative of a tensor field we have

$$D(\boldsymbol{e}_i, \boldsymbol{E}^k \otimes \boldsymbol{e}_j) = \boldsymbol{E}^k \otimes D(\boldsymbol{e}_i, \boldsymbol{e}_j) + D(\boldsymbol{e}_i, \boldsymbol{E}^k) \otimes \boldsymbol{e}_j.$$

But covariant derivative preserves contractions. Contracting the above equality we obtain

$$\boldsymbol{e}_i\left(\boldsymbol{E}^k(\boldsymbol{e}_j)\right) = \boldsymbol{E}^k\left(\sum_a^{1,m}\Gamma^a_{ij}\boldsymbol{e}_a\right) + D(\boldsymbol{e}_i,\boldsymbol{E}^k)(\boldsymbol{e}_j),$$

which gives

$$\boldsymbol{e}_i(\delta^k_j) = \sum_a^{1,m}\Gamma^a_{ij}\boldsymbol{E}^k(\boldsymbol{e}_a) + \boldsymbol{e}_j(D(\boldsymbol{e}_i,\boldsymbol{E}^k)).$$

In turn this implies

$$0 = \sum_a^{1,m}\Gamma^a_{ij}\delta^k_a + (D(\boldsymbol{e}_i,\boldsymbol{E}^k))_j,$$

from which we get

$$(D(\boldsymbol{e}_i,\boldsymbol{E}^k))_j = -\Gamma^k_{ij}.$$

Let us now consider a generic $(0,1)$-tensor $\boldsymbol{\omega} = \sum_i^{1,m}\omega_i\boldsymbol{E}^i$. Then we have

$$\begin{aligned}
(D(\boldsymbol{e}_i,\boldsymbol{\omega}))_k &= \boldsymbol{e}_k(D(\boldsymbol{e}_i,\boldsymbol{\omega}))\\
&= \boldsymbol{e}_k(D(\boldsymbol{e}_i,\sum_j^{1,m}\omega_j\boldsymbol{E}^j))\\
&= \boldsymbol{e}_k(\sum_j^{1,m}\left[\boldsymbol{e}_i(\omega_j)\boldsymbol{E}^j + \omega_j D(\boldsymbol{e}_i,\boldsymbol{E}^j)\right])\\
&= \sum_j^{1,m}\boldsymbol{e}_k(\boldsymbol{e}_i(\omega_j)\boldsymbol{E}^j + \omega_j D(\boldsymbol{e}_i,\boldsymbol{E}^j))\\
&= \sum_j^{1,m}\left(\boldsymbol{e}_i(\omega_j)\boldsymbol{e}_k(\boldsymbol{E}^j) + \omega_j\boldsymbol{e}_k(D(\boldsymbol{e}_i,\boldsymbol{E}^j))\right)\\
&= \sum_j^{1,m}\left(\boldsymbol{e}_i(\omega_j)\delta^j_k - \omega_j\Gamma^j_{ik}\right)\\
&= \boldsymbol{e}_i(\omega_k) - \sum_j^{1,m}\omega_j\Gamma^j_{ik}.
\end{aligned}$$

We are going to denote with a semicolon the covariant derivative also in this case, i.e.

$$\omega_{k;i} = \boldsymbol{e}_i(\omega_k) - \sum_j^{1,m}\omega_j\Gamma^j_{ik}.$$

In a coordinate basis $\{\partial_i\}_{i=1,\dots,m}$ the above becomes

$$\omega_{k;i} = \partial_i\omega_k - \sum_j^{1,m}\omega_j\Gamma^j_{ik}.$$

□