Chapter 17

Lecture 17

We are going to give in this lecture a mathematical structure to the conceptual discussion we presented in the previous one. First we need to specify some further concepts on curves and geodesics, which we are going to use shortly after.

17.1 More about curves on manifolds - 2 -

On a Lorentzian manifold let us consider a curve $\sigma(t)$. We already defined it and called $\dot{\boldsymbol{\sigma}}(t)$ its tangent vector. Since in a Lorentzian manifold we have a notion of scalar product, we can compute the modulus of a general vector, and of a tangent vector of a curve in particular. We remember that since the Lorentzian metric is *not* positive definite, then the modulus square of a vector is not always positive. In particular, following the same procedure that we used in the case of Minkowski space we can give the following definition.

Definition 17.1 (Timelike, spacelike, null vectors on a manifold)
Let us consider a tangent vector $\boldsymbol{v}$ to a Lorentzian manifold $(\mathscr{M}, \mathcal{F}, \langle -, - \rangle)$. $\boldsymbol{v}$ is timelike if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0$, $\boldsymbol{v}$ is spacelike if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$ and $\boldsymbol{v}$ is lightlike or null if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$.

We can now consider a curve σ on $(\mathscr{M}, \mathcal{F}, \langle -, - \rangle)$. A tangent vector to σ at the point $\mathtt{m} = \sigma(t)$ is a tangent vector in $\mathscr{M}_{\mathtt{m}}$.

Definition 17.2 (Local character of a curve at a point)
We will say that the curve σ is timelike (respectively spacelike, null) at a point $\mathtt{m} = \sigma(t)$ if the tangent vector $\dot{\boldsymbol{\sigma}}(t)$ is timelike (respectively spacelike, null).

In general a curve can have different character at different points. But often more interesting are curves which maintain their character along themselves. This are singled out by the following

Definition 17.3 (Global character of a curve)
A curve σ on $(\mathscr{M}, \mathcal{F}, \langle -, - \rangle)$ is called a timelike (respectively spacelike, null) curve, if it is timelike (respectively spacelike, null) at each point.

As we have seen the Lorentzian structure on a manifold gives us the possibility of giving a more detailed classification of vectors and curves. It also gives us the opportunity of introducing metric properties on the manifold. Among the many applications we are especially interested in defining the length of a curve. This can be done as follows.

Definition 17.4 (Length of a curve)
Let σ be a differentiable curve on a Lorentzian manifold $(\mathscr{M}, \mathcal{F}, \langle -,- \rangle)$ and let $t \in [a,b] \subset \mathbb{R}$ be the parameter along σ. The length *of the curve is*

$$\ell(\sigma) = \int_a^b |\langle \dot{\boldsymbol{\sigma}}(t), \dot{\boldsymbol{\sigma}}(t) \rangle|^{1/2} dt.$$

A curve such that $|\dot{\boldsymbol{\sigma}}(\lambda)| \equiv 1$ for all λ is said to be parametrized according to *arc-length* and λ is called a natural parameter.

17.1.1 Geodesics

We already defined in Lecture 14 the concept of an autoparallel curve (but remember the observation in appendix A). In our present setting we are considering the only symmetric connection associated with the Lorentzian metric on the manifold, i.e. we have a covariant derivative that is compatible with the Lorentzian metric.

Definition 17.5 *Given a Lorentzian manifold $(\mathscr{M}, \mathcal{F}, \langle -,- \rangle)$ autoparallel curves with respect to the only symmetric connection compatible with the Lorentzian metric are called* geodesics.

Proposition 17.1 (Character of geodesics)
The character of a geodesic (timelike, spacelike or null), cannot change along the geodesic. In other words if a geodesic is timelike (respectively spacelike, null) at one point it is a timelike (respectively spacelike, null) curve.

Proof:

By definition of a geodesic $\sigma(t)$, the velocity vector field $\dot{\boldsymbol{\sigma}}(t)$ is parallel along the geodesic. On the other hand the character of the geodesic at each is defined by $\langle \dot{\boldsymbol{\sigma}}(t), \dot{\boldsymbol{\sigma}}(t) \rangle$ and since $\dot{\boldsymbol{\sigma}}(t)$ is a parallel vector field along $\sigma(t)$, the compatibility condition (15.1) guarantees that

$$\frac{d}{dt} \langle \dot{\boldsymbol{\sigma}}(t), \dot{\boldsymbol{\sigma}}(t) \rangle = 0,$$

i.e. the character of the geodesic cannot change.

□

We also give the following proposition:

Proposition 17.2 *A geodesic σ is an extremal for the length functional $\ell(\sigma)$.*

17.2 Mathematical formulation of General Relativity

As we already anticipated, we are now going to give a more precise mathematical formulation to the principles discussed in the previous lecture. In particular we are going to model spacetime with a

4-dimensional, Lorentzian, differentiable manifold (connected, oriented and Hausdorff[1]).

This manifold, i.e. spacetime, is also equipped with the only symmetric connection which is compatible with the Lorentzian structure. Using the connection we can also define geodesics on the manifold. The Lorentzian structure assures that each neighborhood in spacetime is diffeomorphically mapped onto a domain of Minkowski spacetime. Minkowski spacetime, i.e. $\mathbb{R}^4$ equipped with Minkowski metric, is also diffeomorphically mapped, locally, onto the manifold through the exponential map. This last fact expresses that locally what happens on the manifold looks like what happens in Minkowski space. This is a realization of the

Equivalence principle in the strong form: *locally, it is always possible to write all (non-gravitational) laws of physics in such a way that they take their special relativistic form.*

The principle of general covariance, i.e. the fact that all reference systems are equivalent for the descriptions of laws of nature, is also naturally embodied in our formulation, precisely in the differential structure of the spacetime manifold.

Let us now consider a free particle (respectively, light ray) in Minkowski space. Its world line is a straight line, inside (respectively, on) the light cone. Note moreover that according to the metric structure of Minkowski space, straight lines (that we interpret as trajectories of free particles (respectively, light rays)) are also timelike (respectively, null) geodesics of Minkowski space. Because of the equivalence principle (in its strong form, now) we know that this law must be locally preserved on the manifold. Using our knowledge of the exponential map, we can thus “project” straight lines from the tangent space (Minkowski space) onto geodesics on the manifold. Thus “free” particle (respectively, light rays) move on timelike (respectively, light-like) geodesics on the manifold. Note that we put the word free between quotation marks: this is because in the case of motion on the manifold the meaning of free will be slightly different. We will understand under the term free motion a motion which is subject only to the gravitational force. This is because, as we will see, the description of the gravitational interaction will be embedded in the Lorentzian manifold structure and represented by the concept of curvature.

To build up evidence in support of the last sentence above we will resort to all our effort in the next lectures. We start in what follows with a preliminary result that can be obtained by considering the *classical* (i.e. non-relativistic) limit for the trajectory of a free particle. In the mathematical framework we have set up, this corresponds to analyze the geodesic equation when the spatial components of the four-velocity vector are small compared to the temporal component. This corresponds to have $|\vec{v}| \ll c$, where c is the speed of light, or, by setting $c \equiv 1$, $|\vec{v}| \ll 1$.

[1]We did not discuss these requirements (and we will not do), but since they appear quoted in many treatment, we just add them here for the sake of completeness.

17.3 The classical limit of the geodesic equation

Let us now consider the equation for a timelike geodesic $\sigma(\lambda)$, in terms of arc length

$$\frac{d^2x^\mu(\lambda)}{d\lambda} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha(\lambda)}{d\lambda}\frac{dx^\beta(\lambda)}{d\lambda} = 0. \tag{17.1}$$

Let us reparametrize the above curve, by using a new parameter σ, i.e. let us consider the reparametrization $\lambda = \lambda(\sigma)$. We are going to call the composition $x^\mu(\lambda(\sigma))$ again as $x^\mu(\sigma)$, not to make the notation too heavy. With this word of caution we then have

$$\begin{aligned}
\frac{dx^\mu}{d\lambda} &= \frac{dx^\mu}{d\sigma}\frac{d\sigma}{d\lambda}\\
\frac{d^2x^\mu}{d\lambda^2} &= \frac{d}{d\lambda}\left(\frac{dx^\mu}{d\sigma}\frac{d\sigma}{d\lambda}\right)\\
&= \frac{d}{d\lambda}\left(\frac{dx^\mu}{d\sigma}\right)\frac{d\sigma}{d\lambda} + \frac{dx^\mu}{d\sigma}\frac{d}{d\lambda}\left(\frac{d\sigma}{d\lambda}\right)\\
&= \frac{d}{d\sigma}\left(\frac{dx^\mu}{d\sigma}\right)\frac{d\sigma}{d\lambda}\frac{d\sigma}{d\lambda} + \frac{dx^\mu}{d\sigma}\frac{d^2\sigma}{d\lambda^2}\\
&= \frac{d^2x^\mu}{d\sigma^2}\left(\frac{d\sigma}{d\lambda}\right)^2 + \frac{dx^\mu}{d\sigma}\frac{d^2\sigma}{d\lambda^2}.
\end{aligned}$$

We can now substitute the two above results into equation (14.1) to get

$$\frac{d^2x^\mu}{d\sigma^2}\left(\frac{d\sigma}{d\lambda}\right)^2 + \frac{dx^\mu}{d\sigma}\frac{d^2\sigma}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\left(\frac{d\sigma}{d\lambda}\right)^2 = 0,$$

which we are going to rewrite as

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} = -\frac{dx^\mu}{d\sigma}\frac{d^2\sigma}{d\lambda^2}\left(\frac{d\sigma}{d\lambda}\right)^{-2}.$$

We are now going to make a special choice for the parameter σ, namely $\sigma = x^0$. Moreover we are going to use an overdot ("$\dot{}$") for the derivative with respect to x^0. The previous equation then becomes

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\frac{d^2\sigma}{d\lambda^2}\left(\frac{d\sigma}{d\lambda}\right)^{-2}\dot{x}^\mu. \tag{17.2}$$

Let us consider the equation resulting when we set $\mu = 0$. Because of our choice of the parameter we have $\dot{x}^\rho \equiv 1$ and $\ddot{x}^\rho \equiv 0$, so that we have

$$0 + \Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\frac{d^2\sigma}{d\lambda^2}\left(\frac{d\sigma}{d\lambda}\right)^{-2};$$

we can use the above equation to substitute in the right-hand side of (17.2) and we get

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\dot{x}^\mu$$

or, in simpler form,

$$\ddot{x}^\mu + \left(\Gamma^\mu_{\alpha\beta} - \Gamma^0_{\alpha\beta}\dot{x}^\mu\right)\dot{x}^\alpha\dot{x}^\beta = 0.$$

The above expression contains four equations, but the one that we get setting $\mu = 0$ is identically satisfied because of the substitution we made. We thus remain with only three equations

$$\ddot{x}^k + \left(\Gamma^k_{\alpha\beta} - \Gamma^0_{\alpha\beta}\dot{x}^k\right)\dot{x}^\alpha\dot{x}^\beta = 0\ ,\ k = 1, 2, 3.$$

We now use the hypothesis that the motion on the curve has a speed which is negligible with respect to the speed of light. This means that[2]

$$x^i \ll x^0 \equiv 1.$$

In first place then, inside the round brackets, the second term is negligible with respect to the first one, i.e.

$$\Gamma^0_{\alpha\beta}\dot{x}^k \ll \Gamma^k_{\alpha\beta},$$

because $\Gamma^0_{\alpha\beta}$ is multiplied by $\dot{x}^k \ll 1$. Thus we have

$$\ddot{x}^k + \Gamma^k_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0\ ,\ k = 1, 2, 3,$$

which we rewrite as[3]

$$\ddot{x}^k + \Gamma^k_{00} + 2\Gamma^k_{0i}\dot{x}^i + \Gamma^k_{jk}\dot{x}^j\dot{x}^k = 0\ ,\ k = 1, 2, 3.$$

Again the third and fourth terms are negligible compared with the first one so that in our approximation we obtain

$$\ddot{x}^k = -\Gamma^k_{00}\ ,\ k = 1, 2, 3. \tag{17.3}$$

This equation has a striking similarity with Newton's equation for a particle on which a force per unit mass $-\Gamma^k_{00}$ is acting. We are going to keep warm this result and discuss again about it in a few lectures, when we will be able to give a better interpretation of Γ^k_{00}.

[2]We set the speed of light equal to 1, i.e. $c \equiv 1$.

[3]We adhere here to the convention that *latin* indices which are repeated are summed on 1, 2 and 3.