Chapter 14

Lecture 14

14.1 More about curves on manifolds - 1 -

In what follows, whenever we will speak of a manifold we will implicitly consider a connection on it, whenever necessary.

14.1.1 Autoparallel curves and the exponential map

Definition 14.1 (Autoparallel)
Let σ be a curve on a manifold $(\mathscr{M}, \mathcal{F})$ such that the vector field $\dot{\boldsymbol{\sigma}}(t)$ tangent to the curve is parallel along the curve, i.e.

$$\frac{D\dot{\boldsymbol{\sigma}}(t)}{dt} = 0.$$

Then σ is an autoparallel curve *on $\mathscr{M}$. Please, see appendix A for a note about this definition.*

Proposition 14.1 (Autoparallelism equation)
Let σ be a curve on $(\mathscr{M}, \mathcal{F})$ and let $(U, \phi) \in \mathcal{F}$ be a chart of $\mathscr{M}$ with coordinate functions $(x_1, \ldots, x_m)$. σ is an autoparallel curve if and only if

$$\frac{d^2x^k}{dt^2} + \sum_{i,j}^{1,m} \Gamma^k_{ij} \frac{dx^i(t)}{dt} \frac{dx^j(t)}{dt} = 0, \quad k = 1, \ldots, m \tag{14.1}$$

where $\phi \circ \sigma(t) = (x^1(t), \ldots, x^m(t))$.

Proof:

In the given coordinates the tangent vector $\dot{\boldsymbol{\sigma}}$ has components:

$$\left(\frac{dx^1(t)}{dt}, \ldots, \frac{dx^m(t)}{dt} \right).$$

But, by the definition above, σ is an autoparallel curve if and only if this tangent vector is a parallel vector field along σ and this is true if and only if it satisfies the m differential equations (11.1) of proposition

Figure 14.1: Exponential of a vector.

11.1, where now $v^l = dx^l(t)/dt$. Thus a curve σ is an autoparallel curve if and only if locally the m equations (14.1) are satisfied.

□

Definition 14.2 *Let* $(\mathscr{M}, \mathcal{F})$ *be a manifold and let* $\mathsf{m} \in \mathscr{M}$ *and* $\boldsymbol{v} \in \mathscr{M}_{\mathsf{m}}$ *be a point of* $\mathscr{M}$ *and a vector tangent to* $\mathscr{M}$ *at* m *respectively. The* exponentiation of $\boldsymbol{v}$ at m *is the point* $\mathsf{p} \in \mathscr{M}$ *which is a unit parameter distance away along the unique autoparallel curve* $\sigma_{\boldsymbol{v}}$ *passing at* m *at* $t = 0$ *and having at* m *tangent vector* $\boldsymbol{v}$*. The* exponential *of the vector* $\boldsymbol{v}$ *at* m *is thus defined as*

$$\exp_{\mathsf{m}}(\boldsymbol{v}) \stackrel{\text{def.}}{=} \sigma_{\boldsymbol{v}}(1)$$

and is a map

$$\exp_{\mathsf{m}} : W \subset \mathscr{M}_{\mathsf{m}} \longrightarrow \mathscr{M},$$

where W *is a neighborhood of* $\mathbf{0}$ *in* $\mathscr{M}_{\mathsf{m}}$.

If k is a constant then of course we have

$$\left.\frac{d}{dt}\right]_{t=0} \sigma_{\boldsymbol{v}}(kt) = k\boldsymbol{v}$$

so that

$$\exp_{\mathtt{m}}(k\boldsymbol{v}) = \sigma_{\boldsymbol{v}}(k).$$

If we interpret k as a parameter we thus have that

$$\sigma_{\boldsymbol{v}}(t) \stackrel{\text{def.}}{=} \exp_{\mathtt{m}}(t\boldsymbol{v})$$

is the only autoparallel curve with $\sigma(0) = \mathtt{m}$ and $\dot{\boldsymbol{\sigma}}(0) = \boldsymbol{v}$. It is defined for small enough t, let us say $-\epsilon < t < \epsilon$, such that $t\boldsymbol{v} \in W$.

Proposition 14.2 *The differential in* $\mathbf{0}$ *of the exponential map at* m *is an isomorphism of* $\mathscr{M}_{\mathsf{m}}$*, in particular*

$$d(\exp_{\mathsf{m}})\rceil_{\mathbf{0}} = \mathbb{I}_{\mathscr{M}_{\mathsf{m}}}.$$

Proof:

To understand $d(\exp_{\mathtt{m}})\rceil_{\mathbf{0}}$ let us remember that according to its definition, the differential is a map between tangent spaces. In this case the tangent spaces are:

1. the tangent space to the tangent space $\mathscr{M}_{\mathtt{m}}$ at the origin $\mathtt{0}$, which as usual we denote with $(\mathscr{M}_{\mathtt{m}})_{\mathbf{0}}$; since $\mathscr{M}_{\mathtt{m}}$ is a vector space, we can up to an isomorphism use the following identification
$$(\mathscr{M}_{\mathtt{m}})_{\mathbf{0}} \approx \mathscr{M}_{\mathtt{m}};$$
2. the tangent space to $\mathscr{M}$ at $\mathtt{m}$, i.e. $\mathscr{M}_{\mathtt{m}}$.

Thus up to an isomorphism

$$d(\exp_{\mathtt{m}})\rceil_{\mathbf{0}} : \mathscr{M}_{\mathtt{m}} \longrightarrow \mathscr{M}_{\mathtt{m}}.$$

We are now interested in the action of $d(\exp_{\mathtt{m}})\rceil_{\mathbf{0}}$ on $\mathscr{M}_{\mathtt{m}}$. To grasp it we can take a curve in $\mathscr{M}_{\mathtt{m}}$ which has $\boldsymbol{v}$ as tangent vector, consider its image under $\exp_{\mathtt{m}}$ and look for the tangent vector of this image (which is a curve on $\mathscr{M}$).
As a curve in $\mathscr{M}_{\mathtt{m}}$ with tangent vector $\boldsymbol{v}$ at the origin $\mathtt{0}$ we can choose the line $l(t) = t\boldsymbol{v}$, $-\epsilon < t < +\epsilon$, for some small enough $\epsilon > 0$. Of course

$$\left.\frac{d}{dt}\right\rceil_{\mathbf{0}} l(t) = \boldsymbol{v}.$$

The exponential then maps this curve into the unique autoparallel curve $\gamma(t) = \exp_{\mathtt{m}}(t\boldsymbol{v})$ passing through $\mathtt{m}$ with tangent vector

$$\left.\frac{d}{dt}\right\rceil_{\mathbf{0}} \gamma(t) = \boldsymbol{v},$$

since $\exp_{\mathtt{m}}$ is defined exactly in this way. Thus

$$d(\exp_{\mathtt{m}})\rceil_{\mathbf{0}} \left(\left.\frac{d}{dt}\right\rceil_{\mathbf{0}} l(t) \right) = \left.\frac{d}{dt}\right\rceil_{\mathbf{0}} \gamma(t) \quad \Rightarrow \quad d(\exp_{\mathtt{m}})\rceil_{\mathbf{0}} (\boldsymbol{v}) = \boldsymbol{v}$$

and this holds $\forall \boldsymbol{v} \in \mathscr{M}_{\mathtt{m}}$, i.e.

$$d(\exp_{\mathtt{m}})\rceil_{\mathbf{0}} = \mathbb{I}_{\mathscr{M}_{\mathtt{m}}}.$$

□

From the above results and the implicit function theorem we conclude that the exponential map is a local diffeomorphism around $\mathbf{0} \in \mathscr{M}_{\mathtt{m}}$ onto a neighborhood $U \subset \mathscr{M}$ of $\mathtt{m}$. It maps lines in the tangent space to autoparallel curves of $\mathscr{M}$ passing through $\mathtt{m}$ and having tangent vector at $\mathtt{m}$ which is the director vector of the line.