Chapter 12

Lecture 12

12.1 Connections on manifolds - 4 -

12.1.1 Component expression of the covariant derivative

We will now compute explicitly $D(\boldsymbol{V}, (D(\boldsymbol{W}, \boldsymbol{Z})))$ in its local form, i.e. when a given basis $\{\boldsymbol{e}_i\}_{i=1,\ldots,m}$ in the tangent space is fixed, so that $\boldsymbol{V} = \sum_i^{1,m} v^i \boldsymbol{e}_i$, $\boldsymbol{W} = \sum_j^{1,m} w^j \boldsymbol{e}_j$ and $\boldsymbol{Z} = \sum_k^{1,m} z^k \boldsymbol{e}_k$. We start with

$$\begin{aligned}
D(\boldsymbol{W}, \boldsymbol{Z}) &= D\left(\sum_j^{1,m} w^j \boldsymbol{e}_j, \sum_k^{1,m} z^k \boldsymbol{e}_k\right) \\
&= \sum_{j,k}^{1,m} D\left(w^j \boldsymbol{e}_j, z^k \boldsymbol{e}_k\right) \\
&= \sum_{j,k}^{1,m} D\left(\boldsymbol{e}_j, z^k \boldsymbol{e}_k\right) w^j \\
&= \sum_{j,k}^{1,m} \left[\boldsymbol{e}_j(z^k) w^j \boldsymbol{e}_k + D(\boldsymbol{e}_j, \boldsymbol{e}_k) w^j z^k\right] \\
&= \sum_{j,k}^{1,m} \boldsymbol{e}_j(z^k) w^j \boldsymbol{e}_k + \sum_{h,j,k}^{1,m} \Gamma^h_{jk} w^j z^k \boldsymbol{e}_h \\
&= \sum_{j,h}^{1,m} w^j \boldsymbol{e}_j(z^h) \boldsymbol{e}_h + \sum_{h,j,k}^{1,m} \Gamma^h_{jk} w^j z^k \boldsymbol{e}_h \\
&= \sum_{h,j}^{1,m} w^j \left(\boldsymbol{e}_j(z^h) + \sum_k^{1,m} \Gamma^h_{jk} z^k\right) \boldsymbol{e}_h \\
&= \sum_{h,j}^{1,m} z^h{}_{;j} w^j \boldsymbol{e}_h,
\end{aligned}$$

so that

$$(D\,(\boldsymbol{W},\boldsymbol{Z}))^i = \sum_k^{1,m} z^i{}_{;k} w^k,$$

where we set up the following

Notation 12.1 (Covariant derivative components)
We will denote the covariant derivative operation on vector/tensor components with a semicolon in the indices, as follows,

$$z^i{}_{;j},$$

where

$$z^i{}_{;j} \stackrel{\text{def.}}{=} \boldsymbol{e}_j(z^i) + \sum_k^{1,m} \Gamma^i_{jk} z^k.$$

If we make the choice of a coordinate basis for $\{\boldsymbol{e}_i\}_{i=1,\ldots,m}$*, i.e.* $\{\boldsymbol{e}_i\}_{i=1,\ldots,m} = \{\partial_i\}_{i=1,\ldots,m}$*, then we have*

$$z^i{}_{;j} = \partial_j z^i + \sum_k^{1,m} \Gamma^i_{jk} z^k.$$

Using a "," to denote usual partial derivatives we see that the notation for covariant derivatives is just a generalization of the one for partial derivatives,

$$z^i{}_{;j} = z^i{}_{,j} + \sum_k^{1,m} \Gamma^i_{jk} z^k.$$

If we define the second covariant derivative of a vector field $\boldsymbol{Z}$ as the covariant derivative of the covariant derivative of $\boldsymbol{Z}$, i.e. $D\,(-, D\,(-,\boldsymbol{Z}))$, then in component notation (i.e. after choosing a coordinate system, as usual, we have

$$z^l{}_{;ij} = z^l{}_{;i;j}.$$

Generalizing we then have

$$\begin{aligned}
D\,(\boldsymbol{V},(D\,(\boldsymbol{W},\boldsymbol{Z}))) &= \sum_{h,j}^{1,m} (D\,(\boldsymbol{W},\boldsymbol{Z}))^h{}_{;j} v^j \boldsymbol{e}_h \\
&= \sum_{h,j}^{1,m} (\sum_k^{1,m} z^h{}_{;k} w^k)_{;j} v^j \boldsymbol{e}_h \\
&= \sum_{h,j,k}^{1,m} (z^h{}_{;k} w^k)_{;j} v^j \boldsymbol{e}_h,
\end{aligned}$$

so that

$$\begin{aligned}
(D\,(\boldsymbol{V},(D\,(\boldsymbol{W},\boldsymbol{Z}))))^i &= \sum_{j,k}^{1,m} (z^i{}_{;k} w^k)_{;j} v^j \\
&= \sum_{j,k}^{1,m} \left[z^i{}_{;k;j} w^k v^j + z^i{}_{;k} w^k{}_{;j} v^j \right] \\
&= \sum_{j,k}^{1,m} \left[z^i{}_{;kj} w^k v^j + z^i{}_{;k} w^k{}_{;j} v^j \right]. \qquad (12.1)
\end{aligned}$$

To gain some more confidence in passing from the intrinsic notation to the component expression, we are going to derive the above result for $D(\boldsymbol{V}, D(\boldsymbol{W}, \boldsymbol{Z}))$ again, by using the properties of the connection, writing as usual the vector fields in a given coordinate basis $\{\partial_i\}_{i=1,\dots,m}$. We have

$$\begin{aligned}
D(\boldsymbol{V}, D(\boldsymbol{W}, \boldsymbol{Z})) &= D\left(\sum_i^{1,m} v^i \partial_i, D\left(\sum_j^{1,m} w^j \partial_j, \sum_k^{1,m} z^k \partial_k\right)\right) \\
&= \sum_i^{1,m} v^i D\left(\partial_i, D\left(\sum_j^{1,m} w^j \partial_j, \sum_k^{1,m} z^k \partial_k\right)\right) \\
&= \sum_i^{1,m} v^i D\left(\partial_i, \sum_{j,k}^{1,m} w^j D\left(\partial_j, z^k \partial_k\right)\right) \\
&= \sum_{i,j}^{1,m} v^i D\left(\partial_i, w^j \sum_k^{1,m} D\left(\partial_j, z^k \partial_k\right)\right) \\
&= \sum_{i,j}^{1,m} v^i D\left(\partial_i, w^j \sum_k^{1,m} \left(\partial_j z^k + z^k D\left(\partial_j, \partial_k\right)\right)\right) \\
&= \sum_{i,j}^{1,m} v^i D\left(\partial_i, w^j \left(\sum_k^{1,m} \partial_j z^k \partial_k + \sum_{k,l}^{1,m} z^k \Gamma^l_{jk} \partial_l\right)\right) \\
&= \sum_{i,j}^{1,m} v^i D\left(\partial_i, w^j \left(\sum_k^{1,m} \partial_j z^k \partial_k + \sum_{k,l}^{1,m} z^l \Gamma^k_{jl} \partial_k\right)\right) \\
&= \sum_{i,j}^{1,m} v^i D\left(\partial_i, w^j \sum_k^{1,m} \left(\partial_j z^k + \sum_l^{1,m} z^l \Gamma^k_{jl}\right) \partial_k\right) \\
&= \sum_{i,j,k}^{1,m} v^i D\left(\partial_i, w^j z^k{}_{;j} \partial_k\right) \\
&= \sum_{i,j,k}^{1,m} v^i \left(\partial_i w^j z^k{}_{;j} \partial_k + w^j D\left(\partial_i, z^k{}_{;j} \partial_k\right)\right) \\
&= \sum_{i,j,k}^{1,m} v^i \left[\partial_i w^j z^k{}_{;j} \partial_k + w^j \left(\partial_i z^k{}_{;j} \partial_k + z^k{}_{;j} D\left(\partial_i, \partial_k\right)\right)\right] \\
&= \sum_{i,j,k}^{1,m} v^i \left[\partial_i w^j z^k{}_{;j} \partial_k + w^j \partial_i z^k{}_{;j} \partial_k + \sum_l^{1,m} w^j z^k{}_{;j} \Gamma^l_{ik} \partial_l\right] \\
&= \sum_{i,j,k}^{1,m} v^i \left[\partial_i w^j z^k{}_{;j} \partial_k + w^j \partial_i z^k{}_{;j} \partial_k + \sum_l^{1,m} w^j z^k{}_{;j} \Gamma^l_{ik} \partial_l + \right. \\
&\qquad \left. + \sum_l^{1,m} w^j z^k{}_{;l} \Gamma^l_{ij} \partial_k - \sum_l^{1,m} w^j z^k{}_{;l} \Gamma^l_{ij} \partial_k\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j,k}^{1,m} v^i \partial_i w^j z^k{}_{;j} \partial_k + \sum_{i,j,k,l}^{1,m} v^i w^j z^k{}_{;l} \Gamma^l_{ij} \partial_k + \\
&\quad + \sum_{i,j,k}^{1,m} v^i w^j \partial_i z^k{}_{;j} \partial_k + \\
&\quad + \sum_{i,j,k,l}^{1,m} v^i w^j z^k{}_{;j} \Gamma^l_{ik} \partial_l - \sum_{i,j,k,l}^{1,m} w^j z^k{}_{;l} \Gamma^l_{ij} \partial_k \\
&= \sum_{i,j,k}^{1,m} v^i z^k{}_{;j} \partial_i w^j \partial_k + \sum_{i,j,k,l}^{1,m} v^i z^k{}_{;j} w^l \Gamma^j_{il} \partial_k + \\
&\quad + \sum_{i,j,k}^{1,m} v^i w^j \partial_i z^k{}_{;j} \partial_k + \\
&\quad + \sum_{i,j,k,l}^{1,m} v^i w^j z^l{}_{;j} \Gamma^k_{il} \partial_k - \sum_{i,j,k,l}^{1,m} w^j z^k{}_{;l} \Gamma^l_{ij} \partial_k \\
&= \sum_{i,j,k}^{1,m} v^i z^k{}_{;j} \left[\partial_i w^j + \sum_{l}^{1,m} w^l \Gamma^j_{il} \right] \partial_k + \\
&\quad + \sum_{i,j,k}^{1,m} v^i w^j \left[\partial_i z^k{}_{;j} + \sum_{l}^{1,m} z^l{}_{;j} \Gamma^k_{il} - \sum_{l}^{1,m} z^k{}_{;l} \Gamma^l_{ij} \right] \partial_k \\
&= \sum_{i,j,k}^{1,m} v^i z^k{}_{;j} w^j{}_{;i} \partial_k + \sum_{i,j,k}^{1,m} v^i w^j (z^k{}_{;j})_{;i} \partial_k \\
&= \sum_{i,j,k}^{1,m} \left[z^k{}_{;ji} w^j v^i + z^k{}_{;j} w^j{}_{;i} v^i \right] \partial_k
\end{aligned}
$$

Note that in the passage before the last one we explicitly see that the covariant derivative is a tensor.

12.2 Curvature - 1 -

12.2.1 Definition of the Riemann tensor and a basic property

Definition 12.1 (Riemann curvature tensor)
The Riemann curvature tensor *is a map*

$$R(-,-)- : \mathcal{V}(\mathscr{M}) \times \mathcal{V}(\mathscr{M}) \times \mathcal{V}(\mathscr{M}) \longrightarrow \mathcal{V}(\mathscr{M}).$$

such that for all triples of vector fields $\boldsymbol{V}$, $\boldsymbol{W}$, $\boldsymbol{Z}$, *the vector field* $R(\boldsymbol{V}, \boldsymbol{W})\boldsymbol{Z}$ *is defined as*

$$R(\boldsymbol{V}, \boldsymbol{W})\boldsymbol{Z} = D(\boldsymbol{V}, (D(\boldsymbol{W}, \boldsymbol{Z}))) - D(\boldsymbol{W}, (D(\boldsymbol{V}, \boldsymbol{Z}))) - D([\boldsymbol{V}, \boldsymbol{W}], \boldsymbol{Z}).$$

If we choose a basis on the tangent space $\{\boldsymbol{e}_i\}_{i=1,\dots,m}$ and let $\{\boldsymbol{E}^j\}_{j=1,\dots,m}$ be its dual basis the components of the Riemann tensor are defined as[1]

$$R^l{}_{ijk} \stackrel{\text{def.}}{=} \boldsymbol{E}^l(R(\boldsymbol{e}_j, \boldsymbol{e}_k)\boldsymbol{e}_i).$$

We remember that in terms of the dual basis $\{\boldsymbol{E}_k\}_{k=1,\dots,m}$ we can also write

$$(D\,(\boldsymbol{W}, \boldsymbol{Z}))^i = \boldsymbol{E}^i(D\,(\boldsymbol{W}, \boldsymbol{Z}))$$

or, of course,

$$(D\,(\boldsymbol{V}, (D\,(\boldsymbol{W}, \boldsymbol{Z}))))^i = \boldsymbol{E}^i(D\,(\boldsymbol{V}, (D\,(\boldsymbol{W}, \boldsymbol{Z})))).$$

We are going to use the above relations in what follows.

Proposition 12.1 (Riemann tensor and covariant derivatives)
The Riemann tensor expresses the non-commutativity of the second covariant derivatives of a vector field.

Proof:

We are going to use the definitions above. From the definition of the Riemann tensor we can extract the components thanks to

$$\begin{aligned}
\boldsymbol{E}^l(R(\boldsymbol{V}, \boldsymbol{W})\boldsymbol{Z}) &= \boldsymbol{E}^l(D\,(\boldsymbol{V}, (D\,(\boldsymbol{W}, \boldsymbol{Z})))) - \boldsymbol{E}^l(D\,(\boldsymbol{W}, (D\,(\boldsymbol{V}, \boldsymbol{Z})))) + \\
&\quad -\boldsymbol{E}^l(D\,([\boldsymbol{V}, \boldsymbol{W}]\,, \boldsymbol{Z})) \\
&= \sum_{j,k}^{1,m} (z^l{}_{;k} w^k)_{;j} v^j - \sum_{j,k}^{1,m} (z^l{}_{;k} v^k)_{;j} w^j + \\
&\quad - \sum_{k}^{1,m} z^l{}_{;k} [\boldsymbol{V}, \boldsymbol{W}]^k \\
&= \sum_{j,k}^{1,m} (z^l{}_{;k;j} w^k v^j + z^l{}_{;k} w^k{}_{;j} v^j) + \\
&\quad - \sum_{j,k}^{1,m} (z^l{}_{;k;j} v^k w^j + z^l{}_{;k} v^k{}_{;j} w^j) + \\
&\quad - \sum_{j,k}^{1,m} z^l{}_{;k} (w^k{}_{;j} v^j - v^k{}_{;j} w^j) \\
&= \sum_{j,k}^{1,m} z^l{}_{;k;j} w^k v^j - \sum_{j,k}^{1,m} z^l{}_{;k;j} v^k w^j \\
&= \sum_{j,k}^{1,m} z^l{}_{;kj} w^k v^j - \sum_{j,k}^{1,m} z^l{}_{;jk} v^j w^k \\
&= \sum_{j,k}^{1,m} \left(z^l{}_{;kj} - z^l{}_{;jk} \right) v^j w^k.
\end{aligned}$$

Of course the left-hand side gives

$$\boldsymbol{E}^l(R(\boldsymbol{V}, \boldsymbol{W})\boldsymbol{Z}) = \sum_{i,j,k}^{1,m} R^l{}_{ijk} v^j w^k z^i$$

[1]Please, pay close attention to the index order in the left and right sides of the coming definition!

so that

$$\sum_{j,k}^{1,m} \left(z_{l;kj} - z^l{}_{;jk} \right) v^j w^k = \sum_{j,k}^{1,m} \left(\sum_i^{1,m} R^l{}_{ijk} z^i \right) v^j w^k$$

and, since $\boldsymbol{V}$ and $\boldsymbol{W}$ are arbitrary vectors,

$$\sum_i^{1,m} R^l{}_{ijk} z_i = z^l{}_{;kj} - z^l{}_{;jk}.$$

$\square$