Chapter 11

Lecture 11

11.1 Tensors - 4 -

11.1.1 A few more concepts about tensors

We are going to make a short interlude to define a couple of additional concepts about tensors $\boldsymbol{T} \in T^r_s(V)$ on a given vector space V of dimension n. Of course, as before our interest is in the application of these concepts to the tensor bundle, which follows naturally. We start defining the tensor algebra over V.

Definition 11.1 (Tensor algebra)
Let us consider the set

$$T_\otimes(V) = \bigoplus_{r,s}^{0,\infty} T^r_s(V).$$

Given $\boldsymbol{T}_1 \in T^r_s(V)$ *and* $\boldsymbol{T}_2 \in T^p_q(V)$ *we will call* $\boldsymbol{T}_1 \otimes \boldsymbol{T}_2$ *the element of* $T^{r+p}_{s+q}(V)$ *that is obtained through the extension by linearity of the map that sends the couple*

$$(v_1 \otimes \ldots \otimes v_r \otimes v_1^* \otimes \ldots \otimes v_s^*, w_1 \otimes \ldots \otimes w_p \otimes w_1^* \otimes \ldots \otimes w_q^*),$$

into

$$v_1 \otimes \ldots \otimes v_r \otimes w_1 \otimes \ldots \otimes w_p \otimes v_1^* \otimes \ldots \otimes v_s^* \otimes w_1^* \otimes \ldots \otimes w_q^*.$$

As usual this map is unique by the universal factorization property. The couple $(T_\otimes(V), \otimes)$ *is an associative algebra over* $\mathbb{F}$*, the* tensor algebra *of* V.

Definition 11.2 (Contractions of a tensor)
Let us consider $(r, s) \in \mathbb{N} \times \mathbb{N}$ *with* $r \geq 1$ *and* $s \geq 1$. $\forall (i, j) \in \mathbb{N} \times \mathbb{N}$ *with* $1 \leq i \leq r$ *and* $1 \leq j \leq s$ *we define the contraction* C^i_j *as the map*

$$C^i_j : T^r_s(V) \longrightarrow T^{r-1}_{s-1}(V)$$

that sends

$$v_1 \otimes \ldots \otimes v_r \otimes v_1^* \otimes \ldots \otimes v_s^*$$

into

$$(v_j^*(v_i)) \cdot v_1 \otimes \ldots \otimes v_{i-1} \otimes v_{i+1} \otimes \ldots \otimes v_r \otimes v_1^* \otimes \ldots \otimes v_{j-1}^* \otimes v_{j+1}^* \otimes \ldots \otimes v_s^* \quad .$$

Remember that in our notation the action of a covector v_j^* on a vector v_i is written as $v_j^*(v_i)$ and, of course, $v_j^*(v_i) \in \mathbb{F}$.

We end this section with an additional intuitive characterization of tensors, that can be useful in some formulations. Let $\boldsymbol{T} \in T_s^r(V)$ be an (r, s)-tensor. We can see it as a multilinear applications that maps r covectors, $\{v_i\}_{i=1,\ldots,r}$, and s vectors, $\{v_j^*\}_{j=1,\ldots,s}$ into an element of $\mathbb{F}$. Let us consider the object

$$\boldsymbol{T}(-, v_2^*, \ldots, v_r^*, v_1, \ldots, v_s).$$

It maps linearly a covector w^* into

$$\boldsymbol{T}(w^*, v_2^*, \ldots, v_r^*, v_1, \ldots, v_s) \in \mathbb{F} \quad ,$$

i.e. it is a linear application from V^* into $\mathbb{F}$. Thus

$$\boldsymbol{T}(-, v_2^*, \ldots, v_r^*, v_1, \ldots, v_s) \in T_0^1(V).$$

Similarly we have, for example,

$$\boldsymbol{T}(-, v_2^*, \ldots, v_r^*, -, v_2, \ldots, v_s) \in T_1^1(V),$$

or

$$\boldsymbol{T}(v_1^*, v_2^*, \ldots, v_r^*, -, -, v_3, \ldots, v_s) \in T_2^0(V),$$

and so on.

11.2 Connections on manifolds - 3 -

11.2.1 Parallel vector fields and parallel translation

Definition 11.3 (Parallel vector field along a curve)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold with connection $D(-,-)$ and let $\sigma(t)$ be a curve on $\mathscr{M}$. A vector field $\boldsymbol{V}(t)$ along σ is parallel *along σ if*

$$\frac{D\boldsymbol{V}}{dt} = 0.$$

Proposition 11.1 (Characterization of parallel vector field)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold of dimension $\dim(\mathscr{M}) = m$ with connection $D(-,-)$. Let $(U, \phi) \in \mathcal{F}$ be a chart for $\mathscr{M}$ with coordinate functions $(x^1, \ldots, x^m)$ and let $\sigma(t) = (x^1(t), \ldots, x^m(t))$ be a curve on $\mathscr{M}$. A vector field $\boldsymbol{V}(t) = \sum_i^{1,m} v^i(t)$ $\partial/\partial x^i$ along σ is parallel *along σ if and only if*

$$\frac{dv^k(t)}{dt} + \sum_{i,j}^{1,m} \frac{dx^i(t)}{dt} \Gamma_{ij}^k v^j(t) = 0 \qquad k = 1, \ldots, m. \tag{11.1}$$

Proposition 11.2 (Existence of parallel vector fields)
Let $\mathscr{M}, \mathscr{F}$ be a manifold and $\sigma(t) = (x^1(t), \ldots, x^m(t))$ be a curve on $\mathscr{M}$. Let $\boldsymbol{v}^0 \in \mathscr{M}_{\sigma(0)}$ be a tangent vector to $\mathscr{M}$ at $\sigma(0)$. There exists one and only one parallel vector field $\boldsymbol{V}$ along σ with $\boldsymbol{V}(\sigma(0)) = \boldsymbol{v}_0$.

Proposition 11.3 (Parallel translation is an isomorphism)
The parallel translation φ along a curve is an isomorphism

$$\varphi : \mathscr{M}_{\sigma(0)} \longrightarrow \mathscr{M}_{\sigma(t)} \quad , \quad \forall t \in [a, b].$$

11.2.2 Extension of covariant derivative to tensors

The connection and its properties have been defined above as operations on vector fields. They can be extended in a natural way to tensors of whatever type. We will give this extension below using also the interpretation of tensors (tensor fields on a manifold) that we have quickly developed at the end of the previous section.

Let us start with a preliminary observation on the operation of covariant derivative that we already know. Given two vector fields, $\boldsymbol{V}$ and $\boldsymbol{W}$, we know that $D(\boldsymbol{V}, \boldsymbol{W})$ is again a vector field. Let us re-express the above sentence by substituting some words with equivalent ones (in particular we are going to substitute *vector field* with $(1,0)$*-tensor field*): given a vector field, $\boldsymbol{V}$, and a $(1,0)$-tensor field, $\boldsymbol{W}$, then $D(\boldsymbol{V}, \boldsymbol{W})$ is again a $(1,0)$-tensor field. Let us then consider $D(-, \boldsymbol{W})$. This is a linear application that associates to each vector field $\boldsymbol{V}$ a $(1,0)$-tensor field, $D(\boldsymbol{V}, \boldsymbol{W})$. Thus $D(-, \boldsymbol{W})$ is a $(1,1)$-tensor field.

From the above considerations the following definition stems:

Definition 11.4 (Covariant derivative of vector fields)
Given a manifold $(\mathscr{M}, \mathcal{F})$ with connection $D(-,-)$ the covariant derivative associated to the given connection is the linear map that associates to each vector field $\boldsymbol{W}$ the $(1,1)$-tensor field $D(-, \boldsymbol{W})$ such that for each vector field $\boldsymbol{V}$, $D(\boldsymbol{V}, \boldsymbol{W})$ is the vector field, which associates to each point $\mathsf{m} \in \mathscr{M}$ the covariant derivative of $\boldsymbol{W}$ in the direction of $\boldsymbol{V}_{\mathsf{m}}$ at m.

The above definition can be extended to any tensor field as follows.

Definition 11.5 (Extension of covariant derivative)
Given a manifold $(\mathscr{M}, \mathcal{F})$ with connection $D(-,-)$ the covariant derivative of a tensor is the map that associates to each (r,s)-tensor field $\boldsymbol{T} \in T^r_s(\mathscr{M})$ the $(r, s+1)$-tensor field $D(-, \boldsymbol{T}) \in T^r_{s+1}(\mathscr{M})$ such that:

1. *$D(-,-)$ is linear;*
2. *$D(-,-)$ commutes with contractions;*
3. *$D(-, \boldsymbol{T}_1 \otimes \boldsymbol{T}_2) = \boldsymbol{T}_1 \otimes D(-, \boldsymbol{T}_2) + D(-, \boldsymbol{T}_1) \otimes \boldsymbol{T}_2$;*
4. *$\forall f \in C^\infty(\mathscr{M})$, $D(-, f) = df$.*
5. *$\forall \boldsymbol{X} \in \mathcal{V}(\mathscr{M}) \cong T^1_0(\mathscr{M})$, then $D(-, \boldsymbol{X})$ is the covariant derivative of the vector field as defined in definition 11.4.*