Chapter 7

Lecture 7

7.1 Tensors - 2 -

7.1.1 Additional properties of tensor product

Proposition 7.1 (Distributive properties of $\otimes$ with respect to $\oplus$)
Given vector spaces U, V, U_i, V_i, $i = 1, \ldots, k$, the following properties hold:

$$\begin{aligned} (U_1 \oplus \ldots \oplus U_k) \otimes V &= U_1 \otimes V \oplus \ldots \oplus U_k \otimes V \\ U \otimes (V_1 \oplus \ldots \oplus V_k) &= U \otimes V_1 \oplus \ldots \oplus U \otimes V_k, \end{aligned} \tag{7.1}$$

where $+$ is the direct sum of vector spaces.

Proposition 7.2 (Basis of tensor product)
Let $\{v_i\}_{i=1,\ldots,m}$ be a basis of V and $\{w_j\}_{j=1,\ldots,n}$ be a basis of W. Then $\{v_i \otimes w_j\}_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$ is basis $U \otimes V$. In particular $\dim(U \otimes V) = \dim(U)\dim(V)$.

Proof:

Let U_i be the subspace of U spanned by u_i and V_j the subspace of V spanned by v_j. By proposition (7.1)

$$U \otimes V = \bigoplus_{i=1,m}^{j=1,\ldots,n} U_i \otimes V_j.$$

At the same time by proposition 6.3 $U_i \otimes V_j$ is a one dimensional vector space spanned by $u_i \otimes v_j$. This completes the proof.

$\square$

Proposition 7.3 (Tensor product and linear applications)
Let

$$L(U^*, V) = \{l : U^* \longrightarrow V, l \quad \text{linear}\}.$$

There exists only one isomorphism,

$$g : U \otimes V \longrightarrow L(U^*, V)$$

such that

$$(g(u,v))(u^*) = u^*(u)v.$$

Proof:

Let us define a bilinear function f,

$$f : U \times V \longrightarrow L(U^*, V) \quad ,$$

such that[1]

$$(f(u,v))(u^*) = u^*(u)v \quad , \quad \forall u \in U \quad , \quad \forall u^* \in U^* \quad , \quad \forall v \in V$$

(remember that $u^*(u) \in \mathbb{F}$). By proposition 6.1 there exists only one g,

$$g : U \otimes V \longrightarrow L(U^*, V)$$

such that, when acting on $u \otimes v$, it gives the same result that f gives when acting on the couple (u, v). This means there exists only one g such that

$$(g(u \otimes v))(u^*) = u^*(u)v \quad .$$

Let us now fix some basis, $\{u_i\}_{i=1,\dots,m}$ in U, $\{u_i^*\}_{i=1,\dots,m}$ in U^* and $\{v_i\}_{i=1,\dots,n}$ in V. Then $\{g(u_i \otimes v_j)\}_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ is a linearly independent set in $L(U^*, V)$. To show this consider a linear combination of these elements

$$\sum_{i=1,m}^{j=1,n} a_{ij} g(u_i \otimes v_j) \quad \text{with} \quad (a_{ij} \in \mathbb{F}, \quad \forall i = 1, \dots, m, \quad \forall j = 1, \dots, n),$$

such that

$$\sum_{i=1,m}^{j=1,n} a_{ij} g(u_i \otimes v_j) = 0.$$

Then we have that

$$\forall k = 1, \dots m, \sum_{i=1,m}^{j=1,n} a_{ij} g(u_i \otimes v_j)(u_k^*) = \sum_{j}^{1,n} a_{kj} v_j = 0,$$

which, since the $\{v_i\}_{i=1,\dots,n}$ are linearly independent, implies

$$a_{kj} = 0 \quad \forall k = 1, \dots, m, \quad \forall j = 1, \dots, n \quad .$$

Since the dimensions of $U \otimes V$ and of $L(U^*, V)$ are the same[2], it follows that g is an isomorphism and for the definition of the universal mapping property it is also unique.

□

Without proof we also give the additional result:

[1] Remember that $u^* \in U^*$ is an application from U into $\mathbb{F}$. Thus $u^*(u) \in \mathbb{F}$. Moreover f is a function from $U \times V$ into $L(U^*, V)$. Thus $f(u, v)$ is a linear map from U^* into V, i.e. $(f(u,v))(u^*) \in V$.

[2] Remember proposition 7.2 and that from the linear algebra course $\dim(L(U^*, V)) = \dim(U^*)\dim(V) = \dim(U)\dim(V)$.

Proposition 7.4 (Tensor product and duals)
Given vector spaces U and V there exists only one isomorphism g

$$g : U^* \otimes V^* \longrightarrow (U \otimes V)^*$$

such that

$$(g(u^* \otimes v^*))(u \otimes v) = u^*(u)v^*(v), \quad \forall u \in U, \forall u^* \in U^*, \forall v \in V, \forall v^* \in V^*.$$

This result can be generalized to r-fold tensor products.

7.1.2 Isomorphism with multilinear transformations

Notation 7.1 *We set up the following notation:*

$$V_r^s \overset{\text{not.}}{=} \overset{1}{V^*} \times \ldots \times \overset{r}{V^*} \times \overset{1}{V} \times \ldots \times \overset{s}{V} .$$

Moreover we set

$$V^s \overset{\text{not.}}{=} V_0^s = \overset{1}{V} \times \ldots \times \overset{s}{V}$$

and

$$V_r \overset{\text{not.}}{=} V_r^0 = \overset{1}{V^*} \times \ldots \times \overset{r}{V^*} .$$

Concerning tensor spaces we set

$$T^r(V) \overset{\text{not.}}{=} \overset{1}{V} \otimes \ldots \otimes \overset{r}{V}$$

and

$$T_s(V) \overset{\text{not.}}{=} \overset{1}{V^*} \otimes \ldots \otimes \overset{r}{V^*} .$$

Then

$$T_s^r(V) \overset{\text{not.}}{=} T^r(V) \otimes T_s(V)$$

with

$$T_0^0 = \mathbb{F}.$$

Proposition 7.5 (Tensor product and linear mappings)
$T_s(V)$ is isomorphic to the space of s-linear mappings from V^s into $\mathbb{F}$.
$T^r(V)$ is isomorphic to the space of r-linear mappings from V_r into $\mathbb{F}$.
$T_s^r(V)$ is isomorphic to the space of (r, s)-linear mappings from V_r^s into $\mathbb{F}$.

Proof:

We prove only the first result using the s-fold generalization of proposition 7.4. We then see that $T_s(V)$ is the dual vector space of $T^s(V)$. But from the universal factorization property generalized to the s-fold tensor product, the dual vector space of $T^s(V)$, which is the linear space of mappings of $T^s(V)$ into $\mathbb{F}$, is isomorphic to the space of s-linear mappings of V^s into $\mathbb{F}$. Analogous proofs can be given in the other cases.

$\square$

7.1.3 Tensors and components

According to the above proposition we can intuitively think a tensor in $T^r_s(V)$ as a multilinear map from V^s_r into $\mathbb{F}$, i.e. as a *black-box* that *eats* r-vectors of V and s covectors of V^* to produce an element of $\mathbb{F}$. We will stick with this representation of a tensor $\boldsymbol{T}$ in $T^r_s(V)$ as a multilinear map in what follows. In this sense, given $\{v_i\}_{i=1,\dots,r}$ in V and given $\{\nu_i\}_{i=1,\dots,s}$ in V^*, we are going to represent the action of $\boldsymbol{T}$ on these sets of vectors and covectors as

$$\boldsymbol{T}(\nu_1, \dots, \nu_s, v_1, \dots, v_r) \in \mathbb{F}.$$

Let us now fix $\{\boldsymbol{e}_i\}_{i=1,\dots,m}$ a basis in V and let $\{\boldsymbol{E}^i\}_{i=1,\dots,m}$ be the corresponding dual basis in V^*. We know from the above results (properly generalized) that

$$\begin{aligned}\big\{\boldsymbol{e}_{i_1} \otimes \dots \otimes \boldsymbol{e}_{i_s} \otimes \boldsymbol{E}^{j_1} \otimes \dots \otimes \boldsymbol{E}^{j_r},& \\ \forall (i_1, \dots, i_s) \,\text{extracted from}\, \{1, \dots, n\}& \\ \text{and}\, \forall (j_1, \dots, j_r) \,\text{extracted from}\, \{1, \dots, n\}\big\}&\end{aligned}$$

is a basis of $T^r_s(V)$. A generic element $\boldsymbol{T} \in T^r_s(V)$ will be written as

$$\sum_{\substack{i_1,\dots,i_r \\ j_1,\dots,j_s}}^{1,n} T^{i_1\dots i_s}_{j_1\dots j_r} \boldsymbol{e}_{i_1} \otimes \dots \otimes \boldsymbol{e}_{i_s} \otimes \boldsymbol{E}^{j_1} \otimes \dots \otimes \boldsymbol{E}^{j_r}$$

in the above basis. According to our interpretation of tensors as multilinear functions, given some vectors $\{\boldsymbol{v}^{(i)}\}_{i=1,\dots,s}$ in V and some *covectors* $\{\boldsymbol{\eta}^{(i)}\}_{i=1,\dots,r}$ in V^* we have that

$$\boldsymbol{T}(\boldsymbol{\eta}^{(1)}, \dots, \boldsymbol{\eta}^{(r)}, \boldsymbol{v}^{(1)}, \dots, \boldsymbol{v}^{(s)}) \in \mathbb{F}. \tag{7.2}$$

Of course we have

$$\boldsymbol{\eta}^{(j)} = \sum_{k_j}^{1,n} \eta^{(j)}_{k_j} \boldsymbol{E}^{k_j}$$

and

$$\boldsymbol{v}^{(j)} = \sum_{h_j}^{1,n} v^{(j)h_j} \boldsymbol{e}_{h_j}.$$

Thus

$$\begin{aligned}&\left(\sum_{\substack{i_1,\dots,i_r \\ j_1,\dots,j_s}}^{1,n} T^{i_1\dots i_s}_{j_1\dots j_r} \bigotimes_{I_a}^{i_1,\dots,i_s} \boldsymbol{e}_{I_a} \otimes \bigotimes_{J_b}^{j_1,\dots,j_s} \boldsymbol{E}^{J_b}\right)(\boldsymbol{\eta}^{(1)}, \dots, \boldsymbol{\eta}^{(r)}, \boldsymbol{v}^{(1)}, \dots, \boldsymbol{v}^{(s)}) = \\ &= \sum_{\substack{i_1,\dots,i_r \\ j_1,\dots,j_s}}^{1,n} T^{i_1\dots i_s}_{j_1\dots j_r} \left[\left(\prod_{I_a}^{i_1,\dots,i_s} \boldsymbol{e}_{I_a}(\boldsymbol{\eta}^{(a)}) \times \prod_{J_b}^{j_1,\dots,j_s} \boldsymbol{E}^{J_b}(\boldsymbol{v}^{(b)})\right)\right] \\ &= \sum_{\substack{i_1,\dots,i_r \\ j_1,\dots,j_s}}^{1,n} T^{i_1\dots i_s}_{j_1\dots j_r} \left[\prod_{I_a}^{i_1,\dots,i_s} \boldsymbol{e}_{I_a} \left(\sum_{k_i}^{1,n} \eta^{(a)}_{k_i} \boldsymbol{E}^{k_i}\right) \times \prod_{J_b}^{j_1,\dots,j_s} \boldsymbol{E}^{J_b} \left(\sum_{h_j}^{1,n} v^{(b)h_j} \boldsymbol{e}_{h_j}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}}^{1,n} T^{i_1\dots i_s}_{j_1\dots j_r} \left[\prod_{I_a}^{i_1,\dots,i_s} \left(\sum_{k_i}^{1,n} \eta^{(a)}_{k_i} \boldsymbol{e}_{I_a}(\boldsymbol{E}^{k_i}) \right) \times \right. \\
&\qquad\qquad \left. \times \prod_{J_b}^{j_1,\dots,j_s} \left(\sum_{h_j}^{1,n} v^{(b)h_j} \boldsymbol{E}^{J_b}(\boldsymbol{e}_{h_j}) \right) \right] \\
&= \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}}^{1,n} T^{i_1\dots i_s}_{j_1\dots j_r} \left[\prod_{I_a}^{i_1,\dots,i_s} \left(\sum_{k_i}^{1,n} \eta^{(a)}_{k_i} \delta^{k_i}_{I_a} \right) \times \prod_{J_b}^{j_1,\dots,j_s} \left(\sum_{h_j}^{1,n} v^{(b)h_j} \delta^{J_b}_{h_j} \right) \right] \\
&= \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}}^{1,n} T^{i_1\dots i_s}_{j_1\dots j_r} \left[\prod_{I_a}^{i_1,\dots,i_s} \eta^{(a)}_{I_a} \times \prod_{J_b}^{j_1,\dots,j_s} v^{(b)J_b} \right] \\
&= \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}}^{1,n} T^{i_1\dots i_s}_{j_1\dots j_r} \eta^{(1)}_{i_1} \cdots \eta^{(s)}_{i_s} v^{(1)j_1} \cdots v^{(r)j_r}.
\end{aligned}
$$

The above expression is the result (7.2) expressed through the components in a given basis. In the same way as $\eta^{(-)}_i$ and $v^{(-)j}$ are the components of the covector $\boldsymbol{\eta}(-)$ and of the vector $\boldsymbol{v}^{(-)}$, respectively, we are going to call $T^{i_1\dots i_s}_{j_1\dots j_r}$ the components of the tensor $\boldsymbol{T}$. In the final expression above the indices i_1, …, i_s and j_1, …, j_r are said to be *saturated* by the vectors $\boldsymbol{v}^{(-)}$ and covectors $\boldsymbol{\eta}(-)$ respectively. The final result above is thus a *scalar*, i.e. an element of $\mathbb{F}$. If not all the indices in the components of a tensor are saturated by vectors or covectors, we get the components of an object which is again a tensor, although of a different kind.

7.2 Synopsis

In the two previous lectures we have defined the tensor product of vector spaces and given the most important properties: we have seen as the *universal factorization property* is a key one for all subsequent derivations: it gives us the possibility of transferring properties of multilinear maps on the cartesian product of vector spaces into properties of tensors and maps on tensors. In what follows we are going to be mainly interested in the (r, s)-fold tensor product of a fixed given vector space V, i.e. on the (r-fold tensor product of V) $\otimes$ (the s-fold tensor product of V^*). We have seen that in this case a tensor $\boldsymbol{T} \in T^r_s(V)$ can be thought as a multilinear map from V^s_r into $\mathbb{F}$. In this situation we have given a meaning to the concept of tensor components when a preferred basis of V is chosen and we have shortly seen the connection of components and tensor calculus.