Chapter 5

Lecture 5

5.1 The group of Lorentz transformations

In the past lecture we derived Lorentz transformations with some physical considerations based on two physical principles, the principle of relativity and the law of propagation of light *in vacuo.* We also see that the obtained transformations had a special property with respect to the interval defined in equation 4.7: in particular they live that interval invariant. We will now derive again Lorentz transformations (in the 2-dimensional case) in a way that makes explicit how the two physical principles discussed previously are mathematically equivalent to the invariance of the relativistic interval.

5.1.1 2-dimensional case

Let us consider the invariant interval defined in our derivation of Lorentz transformations in the previous chapter. In particular let us consider preliminarily the 2-dimensional case, in which the finite invariant interval can be written as

$$s^2 = x^2 - c^2 t^2.$$

In $\mathbb{R}^2$ we take the vector $\boldsymbol{x} = (t, x)$ and we equip the vector space of all these vectors with the pseudo-Euclidean structure defined by the scalar product

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = g_{AB} x^A x^B \quad , \quad A = 1,2 \quad , \quad B = 1,2,$$

where $g_{00} = -1$, $g_{01} = g_{10} = 0$ and $g_{11} = +1$. Requiring the invariance of the interval is tantamount of requiring the invariance of the pseudo-Euclidean structure: we are now interested in determining the general form of a linear transformation Λ such that

$$\boldsymbol{g} = \Lambda^T \boldsymbol{g} \Lambda.$$

From the validity of the above equation we know that the 2×2 matrix Λ is subject to the constraint

$$\det(\boldsymbol{g}) = \det(\Lambda^T \boldsymbol{g} \Lambda) = \det(\Lambda^T) \det(\boldsymbol{g}) \det(\Lambda)$$

which, since $\det(\Lambda) = \det(\Lambda^T)$, gives

$$\det(\Lambda)^2 = 1 \quad \Rightarrow \quad \det(\Lambda) = \epsilon \overset{\text{def.}}{=} \pm 1.$$

Moreover from the invariance of $\boldsymbol{g}$, if we set

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we obtain:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Performing the matrix multiplications on the right hand side we obtain

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c^2 - a^2 & -ab + cd \\ -ab + cd & b^2 - d^2 \end{pmatrix};$$

this gives three equations which, together with the constraint on the determinant

$$\det(\Lambda) = ad - bc = \epsilon,$$

we can rewrite as a system of four equations in four unknowns:

$$\begin{cases} a^2 - c^2 = 1 \\ cd - ab = 0 \\ b^2 - d^2 = 1 \\ ad - bc = \epsilon \end{cases} . \tag{5.1}$$

Note that, of course, the last equation is dependent from the other three. Thus only three parameters can be determined independently, or more precisely, the solution is going to be a one parameter family of transformations. In what follows we will call with capital letters the signs of the four parameters a, b, c, d, so that

$$\begin{matrix} a = A|a| & , & b = B|b| \\ c = C|c| & , & d = D|d| \end{matrix}$$

Let us set some constraints on them, as a preliminary step:

1. from the first equation we see that $a \neq 0$, i.e. $A \neq 0$.
2. from the third equation we see that $d \neq 0$, i.e. $D \neq 0$.
3. thus, for the signs the equations, respectively, imply:

$$\begin{cases} A \neq 0 \\ AB = CD \\ D \neq 0 \\ AD - BC = \epsilon \end{cases} . \tag{5.2}$$

Let us now solve the first equation for a, the third for d and substitute in the second[1]:

$$\begin{cases} a = A\sqrt{1 + c^2} \\ AB|b|\sqrt{1 + c^2} = CD|c|\sqrt{1 + b^2} \\ d = D\sqrt{1 + b^2} \\ ad - bc = \epsilon \end{cases}$$

[1] Square roots are always *arithmetic* i.e. their sign is always positive.

Using the second equation in (5.2) the second equation above can be simplified and squared to obtain $|b| = |c|$ as a solution. This can be rewritten as $b = \eta c$, where $\eta \stackrel{\text{def.}}{=} -1, 0, +1$. Using this relation in the third equation we also find $|a| = |d|$, so that we end up with the system:

$$\begin{cases} a = A\sqrt{1+c^2} \\ b = \eta c \\ |a| = |d| \\ ad - bc = \epsilon \end{cases}$$

Let us now rewrite the last equation in the above system in a different way that we are going to use later on. We have

$$\begin{aligned} ad - bc &= AD|a||d| - BC|b||c| \\ &= ADa^2 - BCc^2 \\ &= AD(1+c^2) - BCc^2 \\ &= (AD - BC)c^2 + AD = \epsilon \end{aligned} \tag{5.3}$$

Case $\eta = 0$.

In this case $B = C = 0$, i.e. $b = c = 0$. Then $a = A$ and $d = D$ and there can be a sign difference between a and d. This is consistent the fourth equation, which exactly gives $AD = \epsilon$. Thus we obtain

$$\Lambda = A \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}.$$

Making the signs appear explicitly we obtain 4 matrix, the identity and four discrete transformations, as follows:

$$\begin{aligned} \text{Identity} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \text{Time reflection} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ \text{Space reflection} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \text{Space time reflection} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Case $\eta \neq 0$.

In this case $B = \pm 1$ and $C = \pm 1$. We can multiply the second equation in the system (5.2), relating the signs, by A and C, since now both are different from zero, to get $AD = BC$, i.e. $AD - BC = 0$. Substituting this identity in (5.3) we obtain again

$$AD = \epsilon.$$

Since $AD = BC$ and $BC = \eta$ we see that $\epsilon = \eta$, so that the fourth equation (5.1) is again a consequence of the other three. We are going to use ϵ in place of η in what follows, i.e. $b = \epsilon c$. As we anticipated the remaining three equations in (5.1) do not allow an unique solution of the system. Let us parametrize the family of solutions using $\beta = c/a$ (remember $a \neq 0$ always). Then we can rewrite the first three equations of (5.1) as

$$\begin{cases} 1 - \beta^2 = a^{-2} \\ \beta = b/d \\ b^2 - d^2 = 1 \end{cases} .$$

This gives

$$\begin{cases} |a| = \left(1 - \beta^2\right)^{-\frac{1}{2}} \\ |b| = |\beta||d| = |c| \\ |d| = |a| \end{cases} ,$$

so that

$$\Lambda = \begin{pmatrix} A\left(1 - \beta^2\right)^{-\frac{1}{2}} & B|\beta|\left(1 - \beta^2\right)^{-\frac{1}{2}} \\ C|\beta|\left(1 - \beta^2\right)^{-\frac{1}{2}} & D\left(1 - \beta^2\right)^{-\frac{1}{2}} \end{pmatrix} .$$

From the above result we factor out the sign of a

$$\Lambda = A \begin{pmatrix} \left(1 - \beta^2\right)^{-\frac{1}{2}} & AB|\beta|\left(1 - \beta^2\right)^{-\frac{1}{2}} \\ AC|\beta|\left(1 - \beta^2\right)^{-\frac{1}{2}} & AD\left(1 - \beta^2\right)^{-\frac{1}{2}} \end{pmatrix} .$$

We can then fix the signs using previous results with the addition that $\mathrm{sign}\,(\beta) = AC$:

$$\begin{cases} A = \epsilon D \\ B = \epsilon C \\ AB = CD \quad \Rightarrow \quad AD = BC \\ \mathrm{sign}\,(\beta) = AC \end{cases} .$$

This gives

$$\Lambda = A \begin{pmatrix} \left(1 - \beta^2\right)^{-\frac{1}{2}} & \epsilon\,\mathrm{sign}\,(\beta)\,|\beta|\left(1 - \beta^2\right)^{-\frac{1}{2}} \\ \mathrm{sign}\,(\beta)\,|\beta|\left(1 - \beta^2\right)^{-\frac{1}{2}} & \epsilon\left(1 - \beta^2\right)^{-\frac{1}{2}} \end{pmatrix}$$

and we can conclude

$$\Lambda = A \begin{pmatrix} \left(1 - \beta^2\right)^{-\frac{1}{2}} & \epsilon\beta\left(1 - \beta^2\right)^{-\frac{1}{2}} \\ \beta\left(1 - \beta^2\right)^{-\frac{1}{2}} & \epsilon\left(1 - \beta^2\right)^{-\frac{1}{2}} \end{pmatrix} .$$

Although this result has been obtained when $\beta \neq 0$, it reproduces for $\beta = 0$ the identity matrix as well as the reflections obtained above. We will adhere to the convention

$$\gamma = \left(1 - \beta^2\right)^{-\frac{1}{2}} .$$

Then the set

$$\left\{ \Lambda \,\middle|\, \Lambda = A \begin{pmatrix} \gamma & \epsilon\gamma\beta \\ \gamma\beta & \epsilon\gamma \end{pmatrix} , A = \pm 1, \epsilon = \pm 1, -1 \leq \beta \leq 1 \right\} ,$$

which equipped with matrix multiplication is a group, the *Lorentz group*, is the invariance group of the metric $\boldsymbol{g}$

5.1.2 A comment about the 4-dimensional case

The results obtained in the 2-dimensional case can be generalized to 4-dimensions, if we consider the symmetry transformations of the 3-dimensional space.

5.2 Synopsis

We have derived in 2 dimensions the group of transformations that leave the interval $x^2 - c^2t^2$ invariant. This group is the Lorentz group. From the form of the Lorentz group and remembering the results obtained in our previous lecture, we see that these transformations can be physically interpreted as space, time and spacetime reflections and as the motions with uniform velocity V in the direction of the unique spatial x-axis. Additional transformations (substantially space rotations and uniform motions in an arbitrary spatial direction) are also part of this group in the more general 4-dimensional case.