Chapter 1

Lecture 1

1.1 Introduction

In this first lecture we will give a short introduction to the topics that we are going to cover in the course. Before that we will make some premises, which are required to understand the main motivations and goals that will justify our forthcoming efforts. We will be slightly *philosophical* in style, with the aim of putting the right emphasis on some crucial points that will be a common ground on which we are going to move in the future.

1.2 A "glimpse" at Mathematical Physics

As the name of the course, *Mathematical Physics*, already says, we are going to develop methods and concepts that pertain to both fields, Mathematics and Physics: an important point to make clear at the very beginning is how concepts are *defined* in this two disciplines.

1.2.1 Operative definition of physical concepts

In physics concepts are always defined by means of an "operational method", or, which is the same, according to an "operative definition". Operative definitions give, at least in principle, to persons at different places and also at different times, the possibility of reproducing a specific concept by performing a sequence of practical operations. These ideas apply in an evident way to the definitions of the units of measurement, for example. Derived concepts also refer to experimental processes or procedures, which can be direct or indirect (at a different degree). In this way results of measurements can be reproduced (again in principle) by anyone, just by reproducing the experimental setup. Through this procedures (experiments) the physicists have at hand a number, which is always a "rational number" (in general many and, nowadays, often a huge amount, of rational numbers). In this perspective, we *could* set up an *empiric* point of view about knowledge: *everything we can really know is everything we can measure*, and to increase our knowledge we have just to wildly increase the number of our measures to cover as many (useful) situations as possible.

Although this can be considered an admissible point of view it is limited in many ways, and even if we would accept this point of view, after making many and many measures of many and many useful and properly operatively defined quantities, that we can find around us or that we can obtain in some properly chosen setups, we could still be questioned about what would be the next quantity that we would like to measure next.

1.2.2 Axiomatic definition of Mathematical Concepts

In mathematics concepts are instead defined through *axioms*. Axioms characterize some (usually elementary and self explanatory) properties of mathematical concepts. Through a logical, rigorously followed procedure, the mathematicians can derive from the axioms further properties of the concepts, and/or give new useful definitions. A crucial requirement for the *consistency* of all the structure is that axioms are supposed to be non-contradictory. This means that if a first result is obtained (proved) in the framework defined by some axioms, a second result that contradicts the first one *must not* be provable (not because of our lack of ability in finding a proof, of course). We can then assume an *axiomatic* point of view about knowledge: *everything we can really know is everything we can derive from the basic axioms (together with the axioms themselves)* and to increase our knowledge we have only to prove more and more properties from our set of basic axioms (that we can chose, wisely, to be as minimal as possible).

This is also a possible perspective about knowledge, it is itself limited in many ways. In particular although mathematical proofs assure us that conclusions are valid whenever premises are, nothing in mathematics can assure us about the *truth* of the axioms. These are just true by definition. On the other hand, in the history of science it has not rarely happened that mathematical concepts have been defined as abstract formalizations of more practical ideas, often connected with everyday practical problems. Even if we forget this connection of mathematical concepts with experience, certainly we cannot disregard the role of mathematics in many useful models of reality. Certainly in these situations the validity of mathematical results as confronted with experience cannot be considered a secondary aspect and the problem of the empirical consequences of axiomatic assumptions becomes a fundamental one.

1.2.3 Some contact points

In the two previous subsections we have shortly analyzed how concepts are defined in two different disciplines, mathematics and physics. Although our analysis has been quite quick, and possibly limited from many points of view, we hope to have given a sufficiently clear account of how in mathematics and physics concepts are developed according to different procedures. Nevertheless in both fields concepts are defined and used to make *predictions*, i.e. to derive further results and, eventually, to define further concepts. Moreover:

1. for both, a mathematician and a physicist, the concepts they respectively define with their proper own procedures are considered *true*;
2. both, a mathematician and a physicist, probably understand under the word *true*, some kind of reproducibility of the properties that they ascribe to a concept.

We consider important also to stress, that independent developments in both fields according to their respective *guidelines* is certainly valuable and should always be pursued. But we are not going to develop this point of view: on the contrary we are going to give some reasons that make an interesting research field the one in which mathematics and physics come into contact: this is mathematical physics. From what we already said, you can already understand that a mathematical physicists is *both*[1] a mathematician and a physicists. To respect the principles of both disciplines, in mathematical physics we have to put the accent on both, the rigorous formulation of the concepts, according to mathematical principles, as well as to their operative relation with experience. We will thus fight on both sides to build up a valuable framework for the interpretation and knowledge of the universe in which we live.

In more detail let us give a quick list of some concrete advantages that can derive from a strong *alliance* between mathematics and physics:

1. the study of new concrete situations from experimental evidences will stimulate the development of new mathematical concepts and tools;

2. at the same time the rigorous derivation of new results in a mathematical theory will furnish new predictions and lead to the discovery of new aspects of reality;

3. analogies and common properties among different, concrete, experimental results can be more easily understood when a theory for these phenomena is formulated in terms of appropriate mathematical models; in other words, the strong synthesis that can be achieved in a mathematical description can act as a unifying principle for many diverse empirical situations;

4. the *aesthetic principle* in the mathematical formulation of physical laws, can be (and has already been in the past) a guiding principle in the interpretation of the physical (or, more generally, scientific) reality.

Especially the last two aspects make of mathematical physics not just a mixture of mathematics and physics but a discipline in its own right, with a strong internal consistency and a deep predictive power. These high end features are even more useful if we remember that the *truth* of every theory is always approximate, in the sense that it is only valid in the limit of measurement uncertainties: this is why a *not to be forgotten* goal of the mathematical physicist is always the stream to go beyond presently established results, to develop more complete, deeper, and beautiful, but, above all, *better proved*, theories.

1.3 Our program in Mathematical Physics

As is possible to guess directly from the few words we have said above, mathematical physics is a very broad subjects, and many topics can be dealt with under its name. For what concern this course we are going to apply the above general reflection to the concepts of *space* and *time*. In particular we will give an outline of the development of the concepts of space and time as described by differential geometry. Although we are going to concentrate mainly on the

[1]Please, note that we say "both ... and ..." and not "none ... nor ...".

description given in general relativity, on the way we will also shortly discuss the concepts of space and time in pre-relativistic and special relativistic physics. The detailed contents of the course will be (possibly) as follows.

1. A short (mainly informal) account of the transition from the study of the dynamics of systems with a finite number of degrees of freedom to the study of the dynamics of systems with an infinite number of degrees of freedom (fields). This will be mainly a translation dictionary of some concepts already familiar from the course of analytical mechanics.

2. Basics of variational principles in many variables.

3. Complements of linear algebra and differential geometry: in particular tensor fields, Riemannian manifolds, a brief account of the most relevant differences between Riemannian and pseudo-Riemannian manifolds.

4. Basic physical concepts about the theory of relativity, both special and general, with a short account about the pre-relativistic ideas about space and time. Formulation of the above concepts in terms of differential geometry.

5. Formulation of general relativity with, possibly, some basics of causal structure.

For a more detailed list, please refer to the table of contents at the beginning of these notes. The material is divided in chapters according to the order of lectures.