# Appendix A

# Parallelism and Autoparallelism

## A.1 Autoparallel curves

We want to briefly discuss the definition of autoparallel curves we gave in Lecture 14. In particular, because of the literal meaning of the word "autoparallel", a better definition would have been that a curve $\sigma(t)$ is autoparallel if the vector field $\dot{\boldsymbol{\sigma}}(t)$, a vector field along the curve, is such that

$$\frac{D\dot{\boldsymbol{\sigma}}(t)}{dt} = \alpha(t)\dot{\boldsymbol{\sigma}}(t)$$

where $\alpha(t)$ is a function on the curve. According to the result of equation (14.1) then the equation for a geodesic would have been

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha(t)}{dt}\frac{dx^\beta(t)}{dt} = \alpha(t)\frac{dx^\mu(t)}{dt}. \tag{A.1}$$

Let us now choose a new parametrization of $\sigma$, in terms of a parameter $s$, such that $t = t(s)$, where $t$ is a strictly monotone function. We are going to give the following definition,

$$\tilde{x}^\mu(s) \quad = \quad x^\mu(t(s)),$$

and quote the following equalities, that can be easily proved:

$$\begin{aligned}
\frac{dx^\mu(t)}{dt} &= \frac{d\tilde{x}^\mu(s)}{ds}\frac{ds}{dt} \\
\frac{d^2x^\mu(t)}{dt^2} &= \frac{d^2\tilde{x}^\mu(s)}{ds^2}\left(\frac{ds(t)}{dt}\right)^2 + \frac{d\tilde{x}^\mu(s)}{ds}\frac{d^2s(t)}{dt^2}.
\end{aligned}$$

The function $s = s(t)$ is the inverse function of $t = t(s)$ which always exists since $t$ is strictly monotone. Substituting the above results in (A.1) we obtain for the geodesic equation

$$\frac{d^2\tilde{x}^\mu(s)}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{d\tilde{x}^\alpha(s)}{ds}\frac{d\tilde{x}^\beta(s)}{ds} = \frac{d\tilde{x}^\mu(s)}{ds}\left[\alpha(t)\frac{ds}{dt} - \frac{d^2s(t)}{dt^2}\right]\left(\frac{ds(t)}{dt}\right)^{-2}.$$

Let us concentrate un the quantity in square brackets:

$$\alpha(t)\frac{ds}{dt} - \frac{d^2s(t)}{dt^2}.$$

We search a function $s(t)$ such that the above quantity vanishes, i.e. such that

$$\alpha(t)\frac{ds}{dt} - \frac{d^2s(t)}{dt^2} = 0.$$

In the above let us set $y(t) = ds(t)/(dt)$ to obtain

$$\frac{dy(t)}{dt} = \alpha(t)y(t),$$

which can be integrated by separation of variables to obtain

$$y(t) = y_0 \exp\left\{\int_{t_0}^{t} \alpha(\tau)d\tau\right\}.$$

The solution exists under very weak conditions on $\alpha(t)$ which are usually satisfied in the cases of interest. From the above result with an additional integration we can determine the function $s(t)$. Thus under non-restrictive condition on $\alpha(t)$ we can always find a reparametrization of an autoparallel curve (or geodesic) such that it satisfies the definition in Lecture 14 with the corresponding equation given by (14.1).

We are going to call a parameter under which (14.1) holds a natural parameter for the autoparallel curve.