Mathematical Physics

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Basics concepts in field theory

From discrete to continuous systems:

1. Lagrangian theory of a system composed by $N + 1$ identical masses and N identical springs; generalized coordinates and derivation of the equations of motions in the Lagrangian formalism; conjugate momenta and dynamical quantities in the Hamiltonian formulation;
2. continuum limit of a system of identical masses and springs in the Lagrangian formulation; continuum limit of the equations of motion;
3. the concept of field; Lagrangian density, Hamiltonian density, momentum density; basics of Lagrangian and Hamiltonian formulation of a field theory;
4. Basics of variational calculus in many dimensions; functionals, functional derivatives, extremals, derivation of Euler-Lagrange equations for fields from a variational principle; interpretation in view of the heuristic transition from discrete to continuous systems;

Complements of linear algebra.

Tensor product of vector spaces:

1. tensor product, "Universal Factorization Property" of the tensor product for bilinear maps, commutativity of the tensor product, associativity of the tensor product, distributivity of the tensor product with respect to the direct sum of vector spaces;
2. basis of the tensor product, some canonical isomorphisms; generalization of the tensor product and of some properties to the case of a finite number of vector spaces; properties of the dual of a tensor product; isomorphism between tensor products and proper spaces of multilinear applications;
3. the concepts of tensor and of tensor components;
4. contractions of tensors.

Pseudo-Euclidean scalar products:

1. pseudo-Euclidean scalar product/metric; signature of metrics; Lorentzian metrics;
2. classification of time-like, space-like and null vectors.

Differential geometry.

Complements:

1. basic definitions in topology;
2. vector bundles and sections;
3. partition of unity;

4. curves on manifolds, tangent vectors;
5. Lie brackets and properties of Lie brackets.

Tensors and tensor fields on manifold:

1. smooth tensor bundles and smooth tensor fields; tangent and cotangent bundles as tensor bundles; coordinate basis of tensor bundles; characterization of tensor fields; characterization of vector fields.

Connections:

1. the idea of parallel transport; connection at a given point of a manifold; connection on a manifold; symmetric connection; coordinate expression of the connection: connection symbols; characterization of symmetric connections;
2. covariant derivative; covariant derivative along a curve: existence and uniqueness; component expression of the covariant derivative and of the covariant derivative along a curve;
3. parallel vector fields; characterization of parallel vector fields; existence of parallel vector fields; parallel translation and parallel transport as isomorphism;
4. extension of covariant derivative to tensors; second covariant derivative;
5. curvature; definition of the Riemann tensor; relation of the Riemann tensor with the second covariant derivatives of a vector field; components of the Riemann tensor in a coordinate basis; symmetry properties of the Riemann tensor; Ricci tensor.
6. Self-parallel curves and the exponential map on manifolds; properties of the differential of the exponential map; exponential map as a local diffeomorphism.

Riemannian and Lorentzian geometry:

1. Riemannian/Lorentzian metric; existence of Riemannian and Lorentzian metrics;

Relativity.

Special relativity:

1. basics about the concepts of space and time in pre-relativistic physics;
2. principle of special relativity and the law of propagation of light *in vacuo*;
3. apparent contradiction between the principle of special relativity and the law of propagation of light *in vacuo*.
4. Einstein ideas about the concepts of space and time; operational definition of simultaneity; consequences of the idea of operational definition of simultaneity and of the principle of special relativity for the laws of change of a reference system;

5. Lorentz transformations and the Lorentz group; derivations of Lorentz transformation laws starting from the special relativity principle and from the law of propagation of light in vacuo; invariant interval; re-derivation of the laws of Lorentz transformations as the symmetry group of Minkowski metric (in 2 dimensions); basics of the generalization to four dimensions.

General Relativity:

1. inertial and non-inertial systems; Einstein lift experiment; the principle of general covariance; the principle of equivalence;
2. Einstein equations *in vacuo* and their derivation from a variational principle;
3. properties of Einstein equations;
4. physical meaning of the metric field: time measurements, space measurements and clock synchronization;
5. the Newtonian limit of General Relativity;
6. conservation properties in classical physics and in special relativity; conservation laws and general covariance; the stress energy tensor and the form of Einstein equations in the presence of sources.